## Appendix: Proofs of computational statements

**Lemma (4.4).** *For $n \geq 3$ and $k \leq n$, the $\mathcal{G}_{n-1}$ invariants $S_{n,k}$ transform under $T$ according to*

$$S_{n,k}(Tu) = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n-k+\ell}{n-k} u_1^{\ell}\, S_{n,k-\ell}.$$

*Proof.* The argument is induction on $n$. Note first that

$$S_{n,0} = 1$$

satisfies the identity trivially for all $n$. At the other extreme,

$$S_{n,n} = 0$$

also satisfies the statement. To see this, examine

$$\sum_{\ell=0}^{n} (-1)^{\ell} \binom{\ell}{0} u_1^{\ell}\, S_{n,n-\ell} = \sum_{\ell=0}^{n} (-1)^{\ell} u_1^{\ell}\, S_{n,n-\ell} = -u_1 \sum_{m=0}^{n-1} (-1)^m u_1^m\, S_{n,n-1-m}.$$

For the final equality, use $S_{n,n} = 0$ and set $\ell = m+1$. By substituting $x$ for the variable $u_1$ that appears explicitly, the sum factors:

$$\sum_{m=0}^{n-1} (-1)^m x^m\, S_{n,n-1-m} = \prod_{k=1}^{n-1} (u_k - x).$$

Consequently, it vanishes when $x = u_1$.

For the base $n = 3$,

$$\begin{aligned} S_{3,1}(Tu) &= -u_1 + (u_2 - u_1) \\ &= (u_1 + u_2) - 3\,u_1 \\ &= S_{3,1} - 3\,u_1 \end{aligned}$$

and

$$\begin{aligned} S_{3,2}(Tu) &= (-u_1)(u_2 - u_1) \\ &= u_1 u_2 - 2\,(u_1^2 + u_1 u_2) + 3\,u_1^2 \\ &= S_{3,2} - 2\,u_1 S_{3,1} + 3\,u_1^2. \end{aligned}$$

To make the inductive step, use the reduction

$$S_{n+1,k} = S_{n,k} + u_n\, S_{n,k-1}$$

and assume the claim holds for $S_{n,k}$ and $S_{n,k-1}$. (Note that the cases $k = n$ and $k = 1$ fall under the scope of the remarks above.) Thus,

$$\begin{aligned} S_{n+1,k}(Tu) &= S_{n,k}(Tu) + (u_n - u_1)\, S_{n,k-1}(Tu) \\ &= \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n-k+\ell}{n-k} u_1^{\ell}\, S_{n,k-\ell} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{m=0}^{k-1} (-1)^m \binom{n-(k-1)+m}{n-(k-1)} u_1^{m+1}\, S_{n,k-1-m} \\
&+ \sum_{p=0}^{k-1} (-1)^p \binom{n-(k-1)+p}{n-(k-1)} u_1^p\, u_n\, S_{n,k-1-p} \\
= {} & S_{n,k} + \sum_{\ell=1}^{k} (-1)^\ell \binom{n-k+\ell}{n-k} u_1^\ell\, S_{n,k-\ell} \\
&+ \sum_{m=0}^{k-1} (-1)^{m+1} \binom{n-k+(m+1)}{n-k+1} u_1^{m+1}\, S_{n,k-(m+1)} \\
&+ u_n S_{n,k-1} + \sum_{p=1}^{k-1} (-1)^p \binom{n+1-k+p}{n+1-k} u_1^p\, u_n\, S_{n,k-p-1}.
\end{aligned}
$$

Setting $m = \ell - 1$ and $p = \ell$ gives

$$
\begin{aligned}
S_{n+1,k}(Tu) = {} & S_{n,k} + u_n\, S_{n,k-1} \\
&+ \sum_{\ell=1}^{k} (-1)^\ell \binom{n-k+\ell}{n-k} u_1^\ell\, S_{n,k-\ell} \\
&+ \sum_{\ell=1}^{k} (-1)^\ell \binom{n-k+\ell}{n-k+1} u_1^\ell\, S_{n,k-\ell} \\
&+ \sum_{\ell=1}^{k-1} (-1)^\ell \binom{n+1-k+\ell}{n+1-k} u_1^\ell\, u_n\, S_{n,k-\ell-1} \\
= {} & S_{n+1,k} + \sum_{\ell=1}^{k} (-1)^\ell \left( \binom{n-k+\ell}{n-k} + \binom{n-k+\ell}{n-k+1} \right) u_1^\ell\, S_{n,k-\ell} \\
&+ \sum_{\ell=1}^{k-1} (-1)^\ell \binom{n+1-k+\ell}{n+1-k} u_1^\ell\, u_n\, S_{n,k-\ell-1} \\
= {} & S_{n+1,k} + \sum_{\ell=1}^{k} (-1)^\ell \binom{n+1-k+\ell}{n+1-k} u_1^\ell\, S_{n,k-\ell} \\
&+ \sum_{\ell=1}^{k-1} (-1)^\ell \binom{n+1-k+\ell}{n+1-k} u_1^\ell\, u_n\, S_{n,k-\ell-1} \\
= {} & S_{n+1,k} + \sum_{\ell=1}^{k-1} (-1)^\ell \binom{n+1-k+\ell}{n+1-k} u_1^\ell \left( S_{n,k-\ell} + u_n\, S_{n,k-\ell-1} \right) \\
&+ (-1)^k \binom{n+1}{n+1-k} u_1^k
\end{aligned}
$$

$$= S_{n+1,k} + \sum_{\ell=1}^{k-1} (-1)^\ell \binom{n+1-k+\ell}{n+1-k} u_1^\ell \, S_{n+1,k-\ell} + (-1)^k \binom{n+1}{n+1-k} u_1^k$$

$$= \sum_{\ell=0}^{k} (-1)^\ell \binom{(n+1)-k+\ell}{(n+1)-k} u_1^\ell \, S_{n+1,k-\ell}.$$

□

**Lemma (4.6).**

$$\sum_{k=0}^{m} (-1)^k \frac{k+1}{(k+3)!\,(m-k)!} = \frac{m+1}{(m+3)!}.$$

*Proof.* Consider the expansion of the generating function

$$\frac{(1-x)^{m+3}}{x^2} = \sum_{\ell=0}^{m+3} (-1)^\ell \binom{m+3}{\ell} x^{\ell-2}$$

$$= x^{-2} - (m+3)\,x^{-1} + \binom{m+3}{2} + \sum_{\ell=3}^{m+3} (-1)^{\ell-2} \binom{m+3}{\ell} x^{\ell-2}$$

$$= x^{-2} - (m+3)\,x^{-1} + \binom{m+3}{2} + \sum_{k=0}^{m} (-1)^{k+1} \binom{m+3}{k+3} x^{k+1}.$$

Now, differentiate and evaluate at $x = 1$:

$$\frac{d}{dx}\left(\frac{(1-x)^{m+3}}{x^2}\right)\Bigg|_{x=1} = -2 + m + 3 + \sum_{k=0}^{m} (-1)^{k+1} (k+1) \binom{m+3}{k+3}$$

$$0 = m + 1 - (m+3)! \sum_{k=0}^{m} (-1)^k \frac{k+1}{(k+3)!\,(m-k)!}.$$

Rearranging this equation yields the desired statement.

□

**Lemma (4.9).**

$$\sum_{k=0}^{m} \frac{k+1}{k+3} \binom{m+2}{k+2} u_1^{m-k} (u_2 - u_1)^{k+3} = \frac{m+1}{m+3} \left(u_2^{m+3} - u_1^{m+3}\right) - u_1 \, u_2 \left(u_2^{m+1} - u_1^{m+1}\right).$$

*Proof.* Letting $u = u_1$ and $v = u_2 - u_1$,

$$\sum_{k=0}^{m} \frac{k+1}{k+3} \binom{m+2}{k+2} u_1^{m-k} (u_2 - u_1)^{k+3} = \frac{1}{m+3} \sum_{k=0}^{m} (k+1) \binom{m+3}{k+3} u^{m-k} \, v^{k+3}$$

$$= \frac{1}{m+3} \left( \sum_{k=0}^{m} (k+4) \binom{m+3}{k+3} u^{(m+3)-(k+3)} \, v^{k+3} - 3 \sum_{k=0}^{m} \binom{m+3}{k+3} u^{(m+3)-(k+3)} \, v^{k+3} \right)$$

$$= \frac{1}{m+3} \left( \sum_{p=3}^{m+3} (p+1) \binom{m+3}{p} u^{(m+3)-p} v^p - 3 \sum_{p=3}^{m+3} \binom{m+3}{p} u^{(m+3)-p} v^p \right)$$

$$= \frac{1}{m+3} \left( \sum_{p=0}^{m+3} (p+1) \binom{m+3}{p} u^{(m+3)-p} v^p - 3 \sum_{p=0}^{m+3} \binom{m+3}{p} u^{(m+3)-p} v^p \right.$$

$$- u^{m+3} - 2\,(m+3)\, u^{m+2}\, v - 3 \binom{m+3}{2} u^{m+1}\, v^2$$

$$\left. + 3\, u^{m+3} + 3\,(m+3)\, u^{m+2}\, v + 3 \binom{m+3}{2} u^{m+1}\, v^2 \right)$$

$$= \frac{1}{m+3} \left( \sum_{p=0}^{m+3} (p+1) \binom{m+3}{p} u^{(m+3)-p} v^p - 3 \sum_{p=0}^{m+3} \binom{m+3}{p} u^{(m+3)-p} v^p \right.$$

$$\left. + 2\, u^{m+3} + (m+3)\, u^{m+2}\, v \right).$$

The second sum amounts to the binomial expansion of $(u+v)^{m+3} = u_2^{m+3}$ while the first sum is the $v$-derivative of $(u+v)^{m+3}\, v$. In explicit terms, note that

$$\sum_{p=0}^{m+3} \binom{m+3}{p} u^{(m+3)-p}\, v^{p+1} = (u+v)^{m+3}\, v.$$

Hence,

$$\frac{\partial}{\partial v} \left( \sum_{p=0}^{m+3} \binom{m+3}{p} u^{(m+3)-p}\, v^{p+1} \right) = \frac{\partial}{\partial v} ((u+v)^{m+3}\, v)$$

$$\sum_{p=0}^{m+3} (p+1) \binom{m+3}{p} u^{(m+3)-p}\, v^p = (u+v)^{m+3} + (m+3)\,(u+v)^{m+2}\, v.$$

Substituting into the expression above and reverting to $u_1$ and $u_2$,

$$\sum_{k=0}^{m} \frac{k+1}{k+3} \binom{m+2}{k+2} u_1^{m-k}\, (u_2 - u_1)^{k+3}$$

$$= \frac{1}{m+3} \Big( u_2^{m+3} + (m+3)\, u_2^{m+2}\, (u_2 - u_1) - 3\, u_2^{m+3} + 2\, u_1^{m+3} + (m+3)\, u_1^{m+2}\, (u_2 - u_1) \Big)$$

$$= \frac{m+1}{m+3} \big( u_2^{m+3} - u_1^{m+3} \big) - u_1\, u_2 \big( u_2^{m+1} - u_1^{m+1} \big).$$

□

**Lemma (4.13).**

$$\sum_{p=0}^{m} (-1)^p \frac{n-p-1}{n-p+1} \binom{m}{p} = \frac{2\,(-1)^{m-1}}{(n+1)\binom{n}{m}}.$$

*Proof.* Let

$$\Lambda_{n,m} = \sum_{p=0}^{m} (-1)^p \, \frac{n-p-1}{n-p+1} \binom{m}{p} \quad \text{and} \quad L_{n,m} = \frac{2\,(-1)^{m-1}}{(n+1)\binom{n}{m}}.$$

From the reduction

$$L_{n,m} = L_{n,m-1} - L_{n-1,m-1},$$

proceed by induction on $n$ and $m$. For the base relative to $m$:

$$\Lambda_{n,1} = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n\,(n+1)} = L_{n,1}.$$

Make the inductive step by verifying that $\Lambda_{n,m}$ admits the same reduction as $L_{n,m}$. Consider

$$\begin{aligned}\Lambda_{n,m-1} - \Lambda_{n-1,m-1} = & \sum_{p=0}^{m-1} (-1)^p \, \frac{n-p-1}{n-p+1} \binom{m-1}{p} \\ & - \sum_{p=0}^{m-1} (-1)^p \, \frac{n-p-2}{n-p} \binom{m-1}{p}.\end{aligned}$$

Shearing the second sum by one term,

$$\begin{aligned}\Lambda_{n,m-1} - \Lambda_{n-1,m-1} = & \sum_{p=0}^{m-1} (-1)^p \, \frac{n-p-1}{n-p+1} \binom{m-1}{p} \\ & - \sum_{p=1}^{m} (-1)^{p-1} \, \frac{n-(p-1)-2}{n-(p-1)} \binom{m-1}{p-1} \\ = & \frac{n-1}{n+1} + \sum_{p=1}^{m-1} (-1)^p \, \frac{n-p-1}{n-p+1} \binom{m}{p} \frac{m-p}{m} \\ & - \sum_{p=1}^{m-1} (-1)^{p-1} \, \frac{n-p-1}{n-p+1} \binom{m}{p} \frac{p}{m} + (-1)^m \, \frac{n-m-1}{n-m+1} \\ = & \frac{n-1}{n+1} + \sum_{p=1}^{m-1} (-1)^p \, \frac{n-p-1}{n-p+1} \binom{m}{p} + (-1)^m \, \frac{n-m-1}{n-m+1} \\ = & \sum_{p=1}^{m} (-1)^p \, \frac{n-p-1}{n-p+1} \binom{m}{p} \\ = & \Lambda_{n,m}.\end{aligned}$$

□