NOTES

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The Wallet Paradox

Kent G. Merryfield, Ngo Viet, and Saleem Watson

1. Introduction. In his book *aha! Gotcha* [1] Martin Gardner gives an intriguing "paradox" involving money and wallets. We found that an analysis of this paradox can serve as an interesting way of utilizing some key concepts in probability. The paradox as related by Gardner is as follows:

> Each of two persons places his wallet on the table. Whoever has the smallest amount of money in his wallet, wins all the money in the other wallet. Each of the players reason as follows: "I may lose what I have but I may also win more than I have. So the game is to my advantage."

Paradoxically, it seems that the game is to the advantage of both players. Of course if one player always carries a larger amount of money than the other player, then he always loses. So we must require that the game be "fair" in some sense. In his analysis of the problem Kraitchik [2] assumes that the amount of money each person carries is uniformly (discretely) distributed between 0 and 100. He then makes a chart of the distribution of money of both players and observes that the distribution is symmetric (with respect to the diagonal) and concludes that there is no advantage. This explanation is considered unsatisfactory by Gardner since it does not explain what is wrong with the reasoning of the players. Indeed, Kraitchik's chart gives a particular example where the game is not to the advantage of either player, but does not address the source of the paradox. In this article we explore the concept of a fair game and in the process we shall resolve the paradox.

2. What are the random variables? A player says "I may lose what I have but I may also win more than I have." This is a true statement for any single trial of the game. However, the inference that "the game is to my advantage" is the source of the apparent paradox, because it does not take into account the *probabilities* of winning and losing. In other words, if the game is played many times, how often does a player win? How often does he lose? And by how much? Indeed, by considering many trials of this game, the enthusiasm of the players for winning should be tempered by the observation that when one loses one typically has more money in one's wallet.

To analyze this game probabilistically we need to know what are the relevant random variables and what are their probability distributions [3]. We are interested in the probability distributions of W_A and W_B, the amount of money that player A and B will win (or lose), respectively. We say that the game is *fair* if the expected value $E(W_A) = 0$ (equivalently $E(W_B) = 0$). To understand W_A and W_B, let X and Y be the random variables representing the amount of money in the wallets of players A and B, respectively. According to the rules of the game W_A is given by

$$W_A(X,Y) = \begin{cases} -X, & \text{if } X > Y \\ Y, & \text{if } X < Y \\ 0, & \text{if } X = Y \end{cases}$$

and $W_B(X,Y) = -W_A(X,Y)$. These expressions make it difficult to calculate $E(W_A)$ and $E(W_B)$, since they depend on both of the distributions of X and Y in a nontrivial way. This is apparently why no quick and simple way of resolving the paradox is available.

We now consider models of the game that intuitively seem fair.

3. A fair game: independent identically distributed X and Y. Perhaps the most natural model for this game is one in which the distributions of the money in each player's wallet are the same; that is, X and Y are independent, identically distributed random variables on some interval $[a, b]$ or $[a, \infty)$, $0 \leq a < b < \infty$. Thus the joint distributions (X, Y) and (Y, X) have the same density function f satisfying $f(x, y) = f(y, x)$. Now, observing that $W_A(Y, X) = W_B(X, Y)$ and exploiting the symmetry in the problem we have

$$E(W_A) = \int_a^b \int_a^b W_A(x,y) f(x,y)\, dy\, dx$$

$$= \int_a^b \int_a^b W_A(y,x) f(y,x)\, dy\, dx = \int_a^b \int_a^b W_B(x,y) f(x,y)\, dy\, dx = E(W_B)$$

where we have made the change of variables $(x, y) \to (y, x)$, whose Jacobian is 1. This combined with the observation that $W_A(X,Y) = -W_B(X,Y)$ shows that $E(W_A) = 0$; and so, by our definition, this is a fair game.

As a concrete example, suppose X and Y are jointly uniformly distributed on the unit square $[0,1] \times [0,1]$. The probability that player A wins y dollars is $1 - x$. In that case $y \in (x, 1]$ with mean equal to $(1 + x)/2$. Player A loses x dollars with probability x. Given that player A carries x dollars in his wallet, the conditional expectation of the amount of money that he will win is

$$E(W_A | X = x) = \left(\frac{1 + x}{2}\right)(1 - x) - x^2 = \frac{1}{2} - \frac{3}{2}x^2.$$

Thus the expected value for W_A is

$$E(W_A) = \int_0^1 E(W_A | X = x)\, dx = \int_0^1 \left(\frac{1}{2} - \frac{3}{2}x^2\right) dx = 0.$$

It is interesting to consider special cases of this formula for the conditional expectation. Since $E(W_A | X = 1) = -1$ and $E(W_A | X = 0) = 1/2$ we see that a player carrying one dollar in his wallet should expect to lose it, whereas a player carrying nothing in his wallet should expect to gain half a dollar (the mean). Interestingly, if a player is carrying half a dollar (the mean) in his wallet, then $E(W_A | X = 1/2) = 1/8$; that is, his expectation of winning is positive.

4. "The game is *not* to my advantage". It may be tempting to think that the game would be fair if we require only that the distributions X and Y have the same mean. But this is not always the case, as we now show. Suppose that X and Y have the joint distribution shown in the following chart.

$X \backslash Y$	0	1
0	$\frac{2}{6}$	$\frac{2}{6}$
$\frac{3}{2}$	$\frac{1}{6}$	$\frac{1}{6}$

For player A, the marginal distribution is $p(0) = 4/6$ and $p(3/2) = 2/6$ and for player B, the marginal distribution is $p(0) = 3/6$ and $p(1) = 3/6$. The mean for player A is $m_A = (0 \times 4/6) + ((3/2) \times (2/6)) = 1/2$. Similarly the mean for player B is $m_B = (0 \times 3/6) + (1 \times 3/6) = 1/2$. But the expected value of player A's winnings is

$$E(W_A) = 0 \times \frac{2}{6} + 1 \times \frac{2}{6} - \frac{3}{2} \times \frac{1}{6} - \frac{3}{2} \times \frac{1}{6} = -\frac{1}{6}$$

This shows that the game is to the advantage of player B.

It turns out that even a smaller mean does not guarantee an advantage in this game. Indeed, replacing 3/2 in the chart by any number in the interval $(1, 3/2)$ yields an example where player A has a smaller mean than that of B. However, player A is still at a disadvantage (that is, $E(W_A) < 0$).

5. Conclusion. The concept of a fair game has to do with repeated trials (and not with any single trial) of a game. So the wallet game is properly understood in the context of the probability distributions of the money in the wallets and the expected values of winning for each player. We have shown that the game is fair if reasonable assumptions are made on these probability distributions (Sections 3) whereas the game is not fair with other assumptions on these distributions (Section 4). Moreover, our analysis may be used to determine whether the game is fair for any given pair of distributions. So in the context of probability, the paradox is resolved.

Some interesting questions remain unanswered about this problem. For instance, if we suppose that the distributions of players A and B are required to have the same means, is there a strategy that player A could adopt to have a winning edge? In other words, is there a preferred distribution (or a winning strategy)?

REFERENCES

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Department of Mathematics
California State University, Long Beach
Long Beach, CA 90840
kmerry@csulb.edu
viet@csulb.edu
saleem@csulb.edu