# Thales' Shadow

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## Introduction

History records that in the 6th century B.C. Thales of Miletus measured the height of the great pyramid at Giza by comparing its shadow to the shadow of his staff **[3]**. But there are differing versions of how he may actually have done this **[5]**. We will do some mathematical detective work to explore which version is more likely.

The earliest version is attributed to Hieronymus (4th century B.C.) by Diogenes (2nd century A.D.), who writes:

> "[Thales]... succeeded in measuring the height of the pyramids by observing the length of the shadow at the moment when a man's shadow is equal to his own height."

Thales here observes that when one object casts a shadow equal to its height, then all objects cast shadows equal to their own heights. In Figure 1, this means that when $H = S$, then $h = s$. This "equal shadow" phenomenon allows one to measure the height of a tall object by measuring the length of its shadow along the ground. But Thales may have used another method. Plutarch (2nd century A.D.) writes: "Although the king of Egypt admired Thales for many things, he particularly liked the way in which he measured the height of the pyramid without any trouble or instrument." Plutarch continues:

> "[Thales] set up a stick at the tip of the shadow cast by the pyramid, and thus having made two triangles by the sun's rays, he showed that the ratio of the pyramid to the stick is the same as the ratio of the respective shadows."

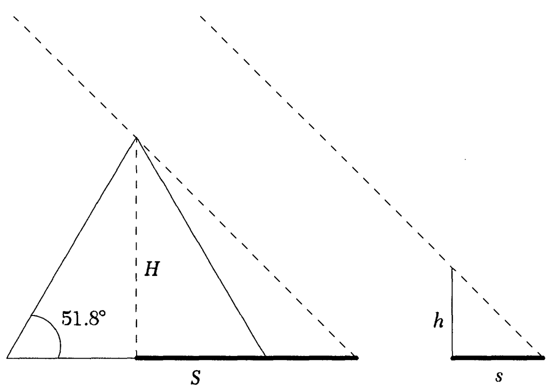



**FIGURE 1**
Shadows of pyramid and stick.

This more general "ratio" method does not require the shadow of the stick to be equal to its length. In FIGURE 1 Thales computes $H$ from the ratio $H/h = S/s$. It is likely that Thales knew about such ratios for legs of right triangles, since the Egyptians also had used such techniques in calculations involving pyramids, calling the method *seked* **[5]**. But neither Thales nor the Egyptians had a general theory of similar triangles. The innovation that Thales makes here is his observation that the sun's rays form a right triangle with the stick and its shadow, so that *seked* can be applied to this abstract right triangle.

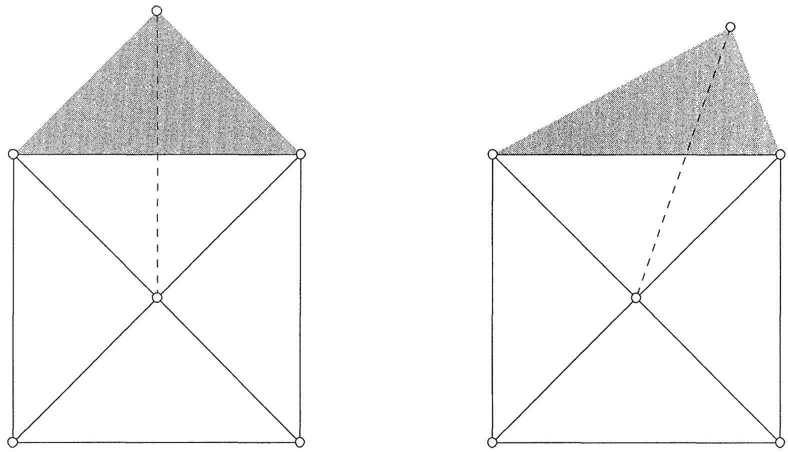
Both the equal shadow method and the ratio method are simple and elegant. The equal shadow method is particularly elegant because it requires no calculations—the length of the shadow is the height of the object being measured. But the shape of the pyramid presents several difficulties, the simplest being that at certain times Thales' staff casts a shadow but the pyramid casts no shadow at all! However, even when the pyramid does cast a shadow, there are still some practical problems to implementing these methods. Before we examine these problems we consider some recorded information about Thales himself, which may give us a hint about his view of theory and applications.

Thales is reputed to have been the first to put geometry on a logical demonstrative basis **[2]**; textbooks on the history of mathematics refer to him as the first mathematician. Evidently, Thales was more interested in logic and proof than in practical matters. According to legend, he was once walking, intently gazing up at the stars, when he fell into a well. A woman with him exclaimed, "How can you tell what is going on in the sky when you can't even see what is lying at your feet?" So it appears that Thales fit the stereotype of a pure mathematician. But this image must be balanced with his association with the very practical Egyptians. In fact, Thales was one of the first Greeks to travel to Egypt, where it is said he learned geometry and also discovered many propositions himself. Egyptian geometry was a tool to serve practical needs which often required extreme precision. Egyptian monuments, still standing millennia later, are witnesses to the accuracy of their builders. Indeed the great pyramid is thought to have been aligned so perfectly north that its minute deviation from true north is attributed by some scientists to continental drift! So, did Thales actually measure the height of the pyramid or did he merely perform a beautiful thought experiment?

## Implementing the ratio method

To use any shadow method one needs to know the length of the shadow as measured from the center of the pyramid. This cannot be done directly since the mass of the pyramid lies between the tip of the shadow and the center. It can be readily done, however, if the shadow is perpendicular to one side of the pyramid as shown in FIGURE 2a. In that case the length of the shadow to the center of the pyramid is simply the length of the shadow along the ground plus the length of half the side of the pyramid. If the shadow is skew, as in FIGURE 2b, then calculating the length of the shadow requires the use of the law of cosines, an idea not available to Thales.

So all Thales needed to do was to visit the pyramid one day and wait for the moment when the shadow of the pyramid is perpendicular to one of the sides. Let us see how this could happen. The pyramid is located at 30° N latitude and since the axis of the earth is tilted 23.5° from the celestial pole, it follows that the pyramid is always located above the plane of the ecliptic. Thus the shadow of the pyramid can never lie on its south side. Let's first consider the case when the shadow lies north of the pyramid. FIGURE 3 shows the situation *at noon*.



(a) (b)

**FIGURE 2**
Top views of pyramid and its shadow.

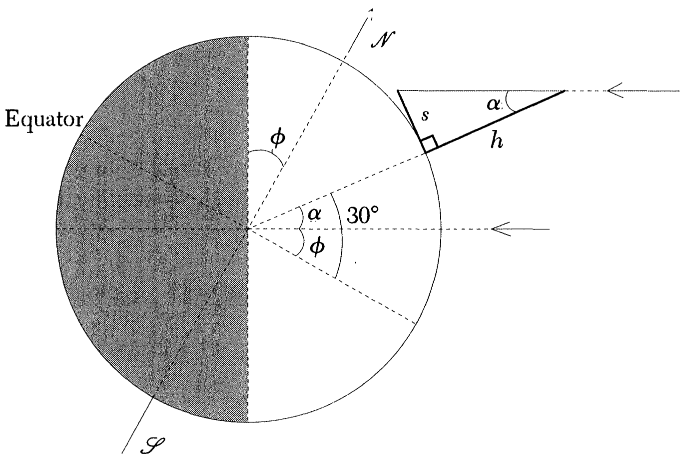



**FIGURE 3**
Shadow of the pyramid at noon.

During the course of a year, as the earth moves around the sun, the angle $\phi$ in FIGURE 3 varies between $-23.5°$ and $23.5°$, so the angle $\alpha = 30° - \phi$ varies between $\alpha = 30° + 23.5° = 53.5°$ at the winter solstice, and $\alpha = 30° - 23.5° = 6.5°$ at the summer solstice (see [4]). The angle $\alpha$ is called the *zenith distance* of the sun because it is the angle formed by the sun and the zenith. From early March to early October, the zenith distance of the sun is small enough at noon so that the pyramid casts no shadow at all—the sun is so high in the sky that all four faces are illuminated. This is because the faces of the pyramid rise at an angle of 51.8° to the horizontal (or $90° - 51.8° = 38.2°$ from the vertical), so whenever the zenith distance of the sun is less than 38.2°, it will shine on all four faces. For the rest of the year, the shadow is perpendicular to the north side once each day. As FIGURE 3 indicates, this occurs at noon—when the line joining the centers of the sun and earth, the polar axis, and the axis of the pyramid all lie in the same plane.

During the spring and summer months, the shadow is also perpendicular to the west face once in the morning, and to the east face once in the afternoon. To see this we need a three-dimensional view of the earth. In FIGURE 4, we have placed

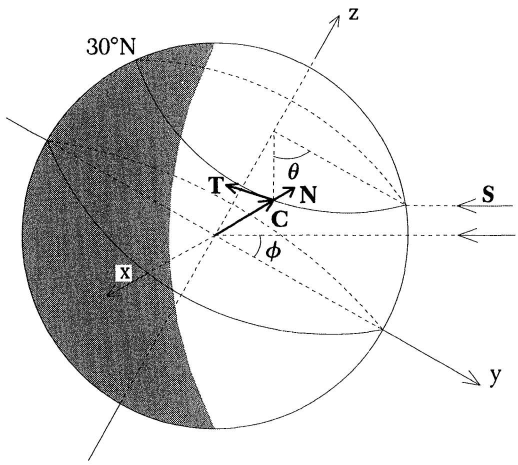



**FIGURE 4**
Pyramid and its shadow—a 3-dimensional view.

coordinate axes with the origin at the center of the earth, the $z$-axis oriented along the polar axis, and the $y$-axis oriented so that the sun lies in the $yz$-plane.

The sun's rays strike the $xy$-plane at the angle $\phi$. Thus the rays of the sun point in the direction

$$\mathbf{S} = -\cos\phi\mathbf{j} - \sin\phi\mathbf{k}.$$

The circle of latitude at 30°N has radius $R\cos 30° = \sqrt{3}R/2$ (where $R$ is the radius of the earth). Thus the position vector of each point on this circle is given by the vector function

$$\mathbf{C}(\theta) = \frac{\sqrt{3}}{2}R\sin\theta\mathbf{i} + \frac{\sqrt{3}}{2}R\cos\theta\mathbf{j} + \frac{1}{2}R\mathbf{k}$$

where $\theta$ is the angle shown in FIGURE 4. Note that $\mathbf{C}(\theta)$ is perpendicular to the sphere of the earth for every $\theta$. The shadow falls east or west when it falls tangent to the latitude circle, that is, in the direction of

$$\mathbf{T}(\theta) = \mathbf{C}'(\theta) = \frac{\sqrt{3}}{2}R\cos\theta\mathbf{i} - \frac{\sqrt{3}}{2}R\sin\theta\mathbf{j}.$$

The axis of the pyramid points in a direction $\mathbf{N}$ normal to the surface of the earth, so at any point on the 30°N latitude circle, we can use $\mathbf{N}(\theta) = \mathbf{C}(\theta)$. The shadow will fall along the tangent to the latitude circle when the tangent $\mathbf{T}$, the normal $\mathbf{N}$, and the rays of the sun $\mathbf{S}$ all lie in the same plane; that is, when $\mathbf{T} \times \mathbf{N} \cdot \mathbf{S} = 0$. Calculating the triple product and simplifying we obtain the condition

$$\cos\theta = \sqrt{3}\tan\phi.$$

Thus the pyramid will cast a shadow perpendicular to its east or west face when it is located at a position corresponding to a value of $\theta$ that satisfies this condition. For instance, at the summer solstice, when $\phi = 23.5°$, we get $\theta \approx \pm 41.14°$. This corresponds to times of about 9:15 AM and 2:45 PM. As the summer progresses, these times will fall earlier in the morning and later in the afternoon, with increasingly longer shadows. At the equinoxes, when $\phi = 0°$, we have $\theta = \pm 90°$, so the shadow will be perpendicular to the west and east faces at 6:00 AM and 6:00 PM respectively —sunrise and sunset on these dates when day and night are equal in duration. It is easy to see that in the fall and winter, when $\phi < 0°$, at no time during daylight hours will the shadow be perpendicular to the east or west face.

## Can the "equal shadow" method be used?

Now suppose Thales was to use the "equal shadow" method. Having found the times when the shadow is perpendicular to the pyramid, he must now find *among these times* "the moment when a man's shadow is equal to his own height." This moment of "equal shadow" can actually happen on at most four days in any given year, as we now show. First let us consider the north face. If the shadow is perpendicular to the north face, then the length of the shadow of an object of height $h$ is $s = h \tan \alpha$ where $\alpha$ is the zenith distance of the sun at noon as in FIGURE 3. Thus we have the "equal shadow" phenomenon when $\alpha = 45°$. During the course of a year $\alpha$ ranges from 6.5° to 53.5° and back to 6.5°, so it appears that $\alpha$ can equal 45° twice. But $\alpha$ changes in increments of approximately $\frac{1}{4}$ degree per day—actually about 94°/365.25 days $\approx 0.26°$ per day. So $\alpha$ may be equal to 45° at most twice. However it is unlikely that $\alpha$ would ever exactly equal 45°, and the error could be as much as $\pm 0.13°$. Since the pyramids are approximately 480 ft high, this would result in an error of approximately 2.3 ft. Given the precision of the Egyptians, this error seems rather high.

Next, for the equal shadow phenomenon to occur at the east or west faces of the pyramid, the angle between **T** and **S** must be 45°. But then

$$\cos 45° = \frac{\mathbf{T} \cdot \mathbf{S}}{|\mathbf{T}||\mathbf{S}|} = \sin \theta \cos \phi,$$

so $\sin \theta = 1/(\sqrt{2} \cos \phi)$. Since the shadow must also be perpendicular to the east or west faces we also have $\cos \theta = \sqrt{3} \tan \phi$, so

$$1 = \sin^2 \theta + \cos^2 \theta = \frac{1}{2 \cos^2 \phi} + 3 \tan^2 \phi.$$

Solving we get $\cos \phi = \pm \sqrt{7/8}$, so $\phi \approx 20.7°$. (The other possible solutions for $\phi$ do not apply here, since $\phi$ takes on values between $-23.5°$ and 23.5°, and negative $\phi$ corresponds to fall and winter when the pyramid does not cast a shadow perpendicular to its east or west face.) Thus there are two days in the year when the "equal shadow" phenomenon occurs on the east or west face—the early spring day and the late fall day when $\phi \approx 20.7°$. Again, just as in the case of the north face, it is unlikely that the "equal shadow" and the "perpendicular shadow" phenomena will coincide precisely on these days, so some imprecision is unavoidable here too.

In any case it seems like a lot of trouble for Thales to hang around the pyramid, possibly for months, waiting for the propitious moment when the equal shadow is also perpendicular to a side. But the King said that Thales measured the height of the pyramids without any trouble. Could Thales have possibly used a different equal-shadow method? One that could work on any day?

## Can the "equal shadow" method be salvaged?

The main problem with implementing the "equal shadow" method is that the measurement of the shadow is obstructed by the mass of the pyramid. Another method Thales could have tried, which also involves "equal shadow," is suggested in the Project Mathematics video [**1**]. The idea is to wait for the shadow of a man to *lengthen* by an amount equal to his height; at the same time the shadow of the pyramid will lengthen by an amount equal to the height of the pyramid (see FIGURE 5). This method is "dynamic" in the sense that it requires observation of the shadow over a period of time, whereas the first method is "static" in that it relies on observing the shadow at just one instant.

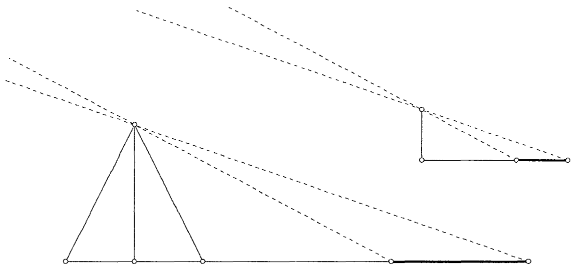


**FIGURE 5**
Lengthening of the shadow.

The important thing here is that this new method appears to avoid the problem of having to measure the shadow to the center of the pyramid. Indeed all measurements take place well away from the pyramid. So let us again see how Thales could practically implement this new method. Thales must visit the pyramid at a time when the tip of the shadow is away from the base of the pyramid and mark the location of the tip of the shadow. He must then wait for the shadow to lengthen. But Thales discovers, to his dismay, that the shadow doesn't simply lengthen as suggested in FIGURE 5, it also moves as shown in FIGURE 6. This presents a new problem, to measure the length of the new longer shadow one must use the center of the pyramid as a reference point. But now we are faced with the same situation as before: the mass of the pyramid obstructs our measurement. So this method isn't really going to work either.

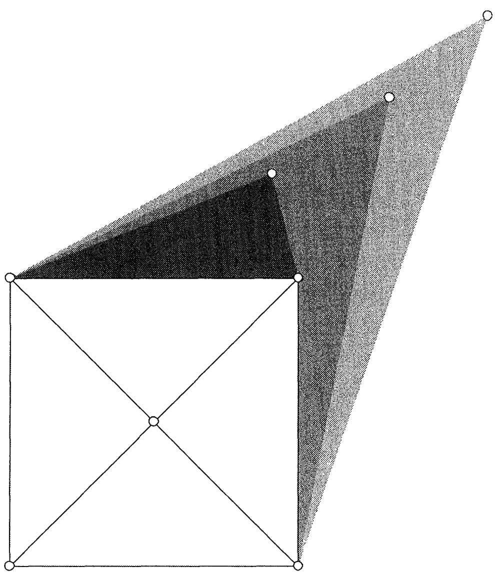


**FIGURE 6**
Lengthening of the shadow—top view.

What if Thales were to simply connect the tip of the original shadow with the tip of the longer shadow and wait for that distance to equal the height? He would now have a distance on the ground equal to the height. This situation is illustrated in FIGURE 7. Triangles $ABC$ and $abc$ are similar, as are triangles $DAB$ and $dab$. From this it is easy to see that $h/H = ab/AB = bc/BC$; since we chose $h = bc$ it follows that $H = BC$. Thus the height of the pyramid can be determined by measuring $BC$ along the ground. This method works, but Thales could not possibly have used it, since he had no knowledge of the proportionality of general similar triangles.

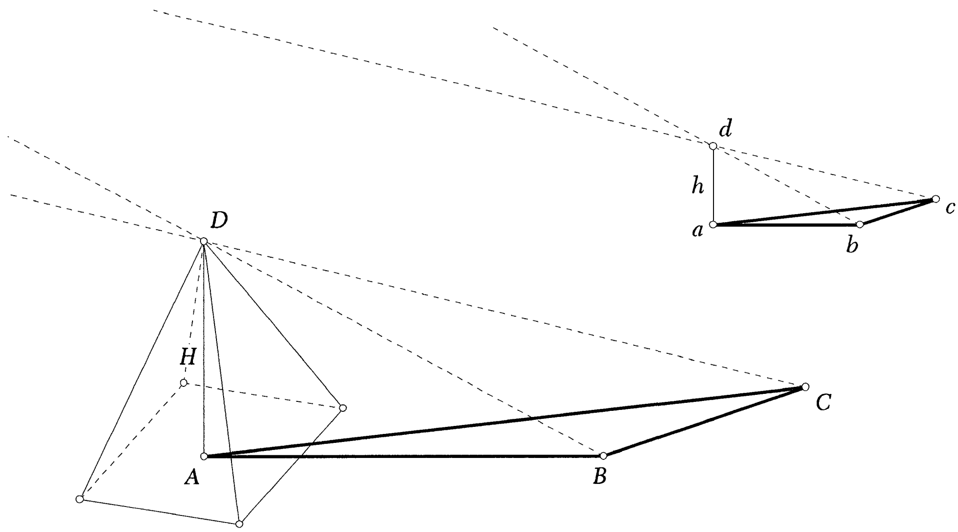



**FIGURE 7**
A method using similar triangles.

## Conclusion

If Thales had used the "equal shadow" method, he would have had to do so on one of only four days in a year, and then only obtained a rough approximation. It seems more likely that he used a "ratio" method. This he would have had an opportunity to do at least once a day for most of the year, and as often as twice on a good day.

Did Thales actually measure the height of the pyramid at all? It is impossible to say for sure, but the *idea* of measuring the height of such a tall object using only its shadow is so beautiful and striking that it overshadows any of its practical applications. This anecdote survives because it encapsulates a great idea that continues to delight and inspire.


### REFERENCES

1. Tom Apostol, *Similarity*, Project Mathematics (videotape), California Institute of Technology, Pasadena, CA, 1990.
2. David Burton, *The History of Mathematics: An Introduction*, 3rd edition, McGraw-Hill, New York, NY, 1997.
3. Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, Oxford, UK, 1972.
4. Lucia Carolyn Harrison, *Daylight, Twilight, Darkness, and Time*, Silver Burdett and Company, New York, NY, 1935.
5. Sir Thomas Heath, *A History of Greek Mathematics*, vol. I, Oxford University Press, Oxford, UK, 1921.