

The Flip-Side of a Lagrange Multiplier Problem

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Introduction. A typical optimization problem in beginning calculus courses is the ‘fencing-a-field’ problem:

Find the dimensions of the rectangular field of maximum area for a fixed perimeter.

There is a natural “flip-side” to this problem:

Find the dimensions of the rectangular field of minimum perimeter for a fixed area.

It is apparent that these problems are related, but what, exactly, is the relationship between them? Do other optimization problems have a flip-side? If so, how does one formulate the flip-side of a given problem?

We give an answer to these questions by considering the more general problem of optimizing a function f of two variables subject to a constraint $g(x, y) = c$ using Lagrange multipliers. As the fencing-a-field problem suggests, the flip-side of a problem involves interchanging the roles of f and g (a process that is meaningful because the Lagrange multiplier condition $\nabla f = \lambda \nabla g$ is symmetric in f and g). In this note we define what is meant by the flip-side of a problem and prove a result that relates an extremum of a problem to an extremum of its flip-side. In following the steps of the proof, students can see how properties of the gradient—in particular the property that the gradient points in the direction of the greatest rate of increase in the values of a function—can be useful visual tools in analyzing optimization problems.

Several articles on Lagrange multipliers have appeared in the CMJ (see for instance [1], [2], [3], [5]), but it seems that the general relationship between a problem and its flip-side (as we call it here) has not been discussed.

The general problem. To better see the relationship between a problem and its flip-side, let’s solve a specific fencing-a-field problem. Suppose the amount of fencing available is 40 units, say. Then the problem is this: Maximize $A(x, y) = xy$ subject to the constraint $P(x, y) = 2x + 2y = 40$. The answer is a square of side 10 and area 100. The flip-side problem is *about fields of area 100*: Minimize P subject to the constraint $A(x, y) = 100$. Again the answer is a square of side 10.

Following this example leads us to the following general situation. Suppose f and g are functions of two variables and f has a local maximum (or minimum) value $m = f(a, b)$ at the point (a, b) subject to the constraint $g(x, y) = c$. The *flip-side* problem is: Does g have a local extremum at (a, b) on the constraint $f(x, y) = m$? And if so, is the extremum a local maximum or minimum?

We show that in general (under appropriate smoothness conditions on f and g) the flip-side problem always has a local extremum at (a, b) , and the type of extremum depends on whether ∇f and ∇g point in the same or opposite directions at (a, b) . We will say that f has a *local maximum point* at (a, b) on the constraint $g(x, y) = c$ if $f(a, b) > f(x, y)$ for all (x, y) on the level set $g(x, y) = c$ in some disc centered at (a, b) .

Theorem. Suppose f and g are smooth functions of two variables, $\nabla f(a, b) \neq 0$, and

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

Let $f(a, b) = c_1$, $g(a, b) = c_2$. If f has a local maximum (minimum) point at (a, b) on the constraint $g(x, y) = c_2$, then the following hold:

1. If $\lambda > 0$, then $g(x, y)$ has a local minimum (maximum) at (a, b) on the constraint $f(x, y) = c_1$.
2. If $\lambda < 0$, then $g(x, y)$ has a local maximum (minimum) at (a, b) on the constraint $f(x, y) = c_1$.

Proof. We prove the result when f has a local maximum at (a, b) . Since

$$\nabla f(a, b) \neq 0 \quad \text{and} \quad \nabla g(a, b) \neq 0$$

the level sets $f(x, y) = c_1$ and $g(x, y) = c_2$ are smooth curves with nonvanishing tangent vectors in some disc centered at (a, b) [1]. Let's call these curves γ_f and γ_g , respectively, as in Figure 1(a). With this terminology, the hypothesis states that f has a local maximum value $c_1 = f(a, b)$ on γ_g . So we can find a small enough disc D on which γ_f and γ_g are smooth curves and $f(x, y) < c_1$ at all other points of γ_g inside D .

Now $f(x, y) > c_1$ on one side of γ_f and $f(x, y) < c_1$ on the other side in D . This is because if $f(x, y) > c_1$ on both sides, then $c_1 = f(a, b)$ is the minimum value of f in D , and so $\nabla f(a, b) = 0$, contradicting the hypothesis. Thus the intersection of the sets

$$M_f = \{(x, y) : f(x, y) \geq c_1\} \quad \text{and} \quad L_f = \{(x, y) : f(x, y) \leq c_1\}$$

in D is γ_f as in Figure 1(a). We have used the letters M and L to indicate the sets where f is "more than" and "less than" (or equal to) c_1 , respectively. Of course g has the same properties as f , so in the same way we define M_g and L_g with respect to c_2 .

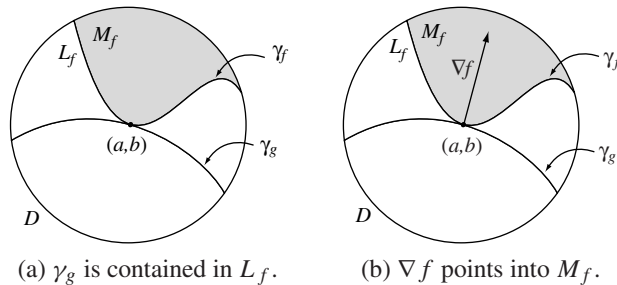
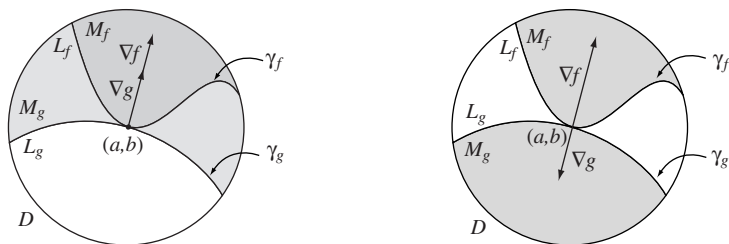


Figure 1. f is larger than c_1 on one side of γ_f and smaller on the other.

Since f attains its maximum value on γ_g at (a, b) , it follows that the values of f on γ_g are all less than or equal to c_1 , that is, $\gamma_g \subset L_f$. See Figure 1(a). (This implies that γ_g is on one side of γ_f , so these curves do not cross in D .) Because the gradient always points in the direction of greatest increase, it follows that $\nabla f(a, b)$ points into M_f (Figure 1(b)).

1. If $\lambda > 0$ then $\nabla f(a, b)$ and $\nabla g(a, b)$ point in the same direction, so we must have $M_f \subset M_g$ as in Figure 2(a). In this case $\gamma_f \subset M_g$, that is, the values of g



(a) $\nabla f, \nabla g$ point in same direction. (b) $\nabla f, \nabla g$ point in opposite directions.

Figure 2. ∇g tells us on which side of γ_g is M_g

on γ_f are all greater than or equal to c_2 . Thus g has the local minimum value $g(a, b) = c_2$ on γ_f .

- If $\lambda < 0$ then $\nabla f(a, b)$ and $\nabla g(a, b)$ point in opposite directions. So we must have $M_g \subset L_f$ as in Figure 2(b). In this case $\gamma_f \subset L_g$, that is, the values of g on γ_f are all less than or equal to c_2 . Thus g has the local maximum value $g(a, b) = c_2$ on γ_f . ■

Moving a river and other flip-side problems. Once a constrained optimization problem has been solved, we can use the theorem to state and solve the flip-side problem. For an applied problem, it is interesting to consider the physical interpretation of the flip-side. For the fencing-the-field problem we want to maximize *area*, for the flip-side we want to minimize *perimeter*. We consider two other common first-semester calculus problems.

The milkmaid problem [2] asks for the minimum distance a milkmaid needs to walk from her home to fetch water from a river and take it to the barn. Specifically, suppose her home is at $(-3, 0)$, the barn at $(3, 0)$, and the river is the line $R(x, y) = 100$, where $R(x, y) = 16x + 15y$. To walk to a point (x, y) and then to the barn the maid travels a distance $d(x, y) = \sqrt{(x + 3)^2 + y^2} + \sqrt{(x - 3)^2 + y^2}$. The problem can now be stated as follows: Minimize d subject to the constraint $R(x, y) = 100$. Using Lagrange multipliers we find that $\lambda > 0$ and the minimum is $d(4, \frac{12}{5}) = 10$. By our theorem the flip-side problem is to maximize R subject to the constraint $d(x, y) = 10$; moreover, the local maximum value is $R(4, \frac{12}{5}) = 100$. We can interpret the flip-side problem as follows: If the maid insists that she will walk a distance of *exactly* 10, then we must move the river for her! That is, we must find the maximum value of c so that she can *just* reach the river $R(x, y) = c$ and then walk to the barn, travelling a total distance of 10.

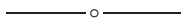
Another ubiquitous problem in first semester calculus courses is the ladder-around-the-corner problem: Find the length of the longest ladder that can go around a rectangular corner with hallways of fixed widths [3]. Using our theorem it is easy to state and solve the flip-side of this problem, but what physical quantity are we actually minimizing in the flip-side problem?

In general, every problem of the type we describe here has a flip-side. For applied problems it is interesting to try and find meaning for the quantities being optimized in the flip-side problem.

References

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Another Proof for the p -series Test

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It is well known that the p -series is $1 + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots$ converges for $p > 1$ and diverges for $p \leq 1$. In standard calculus textbooks (such as [3] and [4]), this is usually shown using the integral test. In this note, we provide an alternative proof of the convergence of the p -series without using the integral test. In fact, our proof is an extension of the nice result given by Cohen and Knight [2].

We begin by giving the following estimate for the partial sum of a p -series:

Lemma. Let $s_n(p)$ be the n th partial sum of the p -series $\sum_{k=1}^{\infty} 1/k^p$.

(a) For $p > 0$,

$$1 - \frac{1}{2^p} + \frac{2}{2^p}s_n(p) < s_{2n}(p) < 1 + \frac{2}{2^p}s_n(p),$$

(b) For $p < 0$,

$$1 + \frac{2}{2^p}s_n(p) < s_{2n}(p) < 1 - \frac{1}{2^p} + \frac{2}{2^p}s_n(p).$$

Proof. As $s_n(p)$ is the n th partial sum,

$$\begin{aligned} s_{2n}(p) &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{(2n)^p} \\ &= 1 + \left[\frac{1}{2^p} + \frac{1}{4^p} + \dots + \frac{1}{(2n)^p} \right] + \left[\frac{1}{3^p} + \frac{1}{5^p} + \dots + \frac{1}{(2n-1)^p} \right]. \end{aligned}$$

For $p > 0$,

$$s_{2n}(p) > 1 + \frac{1}{2^p}s_n(p) + \left[\frac{1}{4^p} + \frac{1}{6^p} + \dots + \frac{1}{(2n)^p} \right].$$

Thus,

$$s_{2n}(p) > 1 + \frac{1}{2^p}s_n(p) - \frac{1}{2^p} + \frac{1}{2^p}s_n(p) = 1 - \frac{1}{2^p} + \frac{2}{2^p}s_n(p).$$