Math 760 2-28

Th^m (Hartshorn)
If M is a closed orientable 3-manifold containing an embedded, closed incomp. surface F, the $d(\Sigma) \leq 2g(F)$ where Σ is any Heegaard Surface for M.

By Perelman's Geometrization Theoren
Every closed orientable 3-manifold is
1. Reducible
2. Contains an incompressible torus
3. is small seifert fibered ← will study more
   or
4. Is Hyperbolic (supports a complete metric of constant scalar curvature -1) will study ~~some~~ a little.

Cor: If M addmitts a Heegaard ~~splitting~~ Surface Σ s.t. $d(\Sigma) \geq 3$, then M is small seifert fibered or Hyperbolic.

Th^m (Scharlemann & Tomova)
Let M be a closed, orientable, irreducible 3-manifold with Heegaard Surfaces P and Q. Then either Q is isotopic to (a possibly stabelized) copy of P or $d(P) < 2g(Q)$

Corr: If P is a H. S. for M with $d(P) > 2g(P)$ then P is the unique minimal genus H. S. for M.

Fact: If M is small ~~stet~~ Seifert fibered, then there exists a H.S. for M of genus less than or equal to 2.

Corr] If M has a H.S. of distance greater than or equal to 4, then M is hyperbolic.

Intro to Dehn Surgery

M is a closed, ~~compt~~ orientable 3-manifold.
$L = L_1 \cup \dots \cup L_n \subset M$ is an embedded link
Let N_i be the closed reg. nbh of L_i
Let $\alpha_i \subset \partial N_i$ be some ess simple closed curve.

$$M^* = (M - (\mathring{N}_1 \cup \dots \cup \mathring{N}_n)) \cup_h (N_1 \cup \dots \cup N_n)$$

where h is the union of homeomorphisms $h_i : \partial N_i \to \partial N_i$ taking a meridian of N_i to α_i.

Claim: The homeomorphism class of M^* is well-defined.

The issue: The isotopy class of h is not well-defined.

Let D_i be ~~the~~ a meridian disk for N_i
Let $D_i \times [-\varepsilon, \varepsilon]$ be a regular nbh of D_i in N_i
Form M^* in stages by first attaching

$$\bigcup_i D_i \times [-\varepsilon, \varepsilon] \text{ to } M - (\mathring{N}_1 \cup \dots \cup \mathring{N}_n)$$

by gluing $\partial D_i \times [-\varepsilon, \varepsilon]$ to ~~α_i~~ a regular nbh of α_i to form M'

So, $M^* = M' \cup \left(\bigcup_{i=1}^{n} B^3_i \right)$

Since there is a unique way of attaching 3-balls to two-sphere boundary components, M^* is well defined.

Surgery Instructions in S^3.

- Each component L_i of an <u>oriented</u> link
- L in S^3 has a prefered framing:
  In particlular there is a unique longitude in $\partial(\eta(L_i))$ that is homologically trivial in $S^3 - L$.

Ex: Prove this using Mayer-Vieterus

Call this orented logitude λ_i (oriented in the same direction as L_i)

Let μ_i be the oriented meridian of $\partial(\eta(L_i))$ having linking number $+1$ with L_i

Hence any ~~homology~~ homotopy class of curve in ∂N_i can be specified by
$$q_i = a_i \lambda_i + b_i \mu_i$$

Hence, the isotopy class of L and the ratio $\frac{b_i}{a_i}$ for each component of L determines the homeomorphism class of M^*

<u>Ex]</u> Show that the choices made above are in fact independant of the orientation of L.

Example: Dehn Surgery on the unknot with slope b/a gives $M^* \cong L(b,a)$.

In particular, $\frac{b}{a} = 0 \Rightarrow M^* \cong S^2 \times S^1$

$\frac{b}{a} = \pm 1/n \Rightarrow M^* \cong S^3$

Big Theorem: Every closed, orientable 3-manifold can be obtained by Dehn Surgery on a link in S^3.

Dehn Surgery on 3-manifolds is equivalent to rational tangle replacement for knots.

Let $K \subset S^3$ and γ be an embedded arc in S^3 s.t. ~~[illegible]~~ $K \cap \gamma = \partial\gamma$. Remove $\eta(\gamma)$ and replace it with a different rational tangle to create K^*.