Grad. Knot Theory Day 1

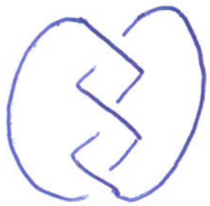
Outline
- Review Syllabus
- Introduction to knot theory
  - Concept of Mathematical model
  - Intuitive def. of knot
  - Intuitive def. of knot equavalence
  - Goals of knot theory
  - Concept of knot invariant
  - Concept of connected sum
  - Old problems in knot theory.

* Concept of Mathematical Model
- Knot theory seeks to model the real-world phenomenon of knots in string, rope, DNA, etc.
- A model is only as "good" or "bad" as its ability to model the real world situation is "good" or "bad."
- Given definitions of "knot" and "equivalence of knots" mathematics can only tell us about those definitions.

Intuitive def. of knot:

* A **knot** is a loop in 3-space.

Ex



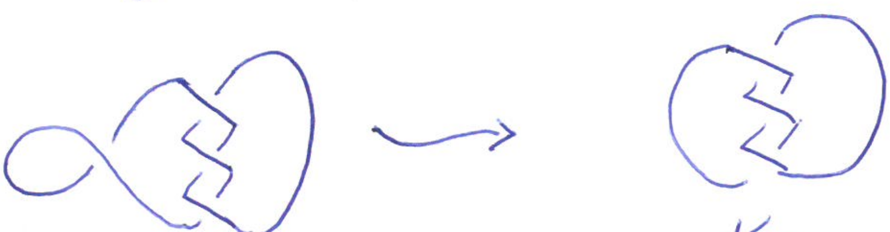
A trefoil

A figure 8 knot

Intuitive def. of knot equivalence

Two knots K_1 and K_2 are **equivalent** if one can be tangled, untangled, bent or stretched to coinside exactly with the other. (no cutting or glueing is allowed).

Ex



K_2

K_1

K_1 is equivalent to K_2.

* Developing definitions for knot and knot equivalence for a "good" model is challenging

* Lets learn from some initial failures.

Recall: If (M, d_M) and (N, d_N) are metric spaces, then $f: M \to N$ is continuous if $\forall x \in M$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $d_M(x,y) < \delta$, then $d_N(f(x), f(y)) < \varepsilon$.

Ex| $S^1 = \{ e^{i\theta} \in \mathbb{C} \mid 0 \leq \theta \leq 2\pi \}$

$$d_{S^1}(e^{i\theta}, e^{i\gamma}) = \min\{|\theta - \gamma|, 2\pi - |\theta - \gamma|\}$$

(i.e. arc length between two points.

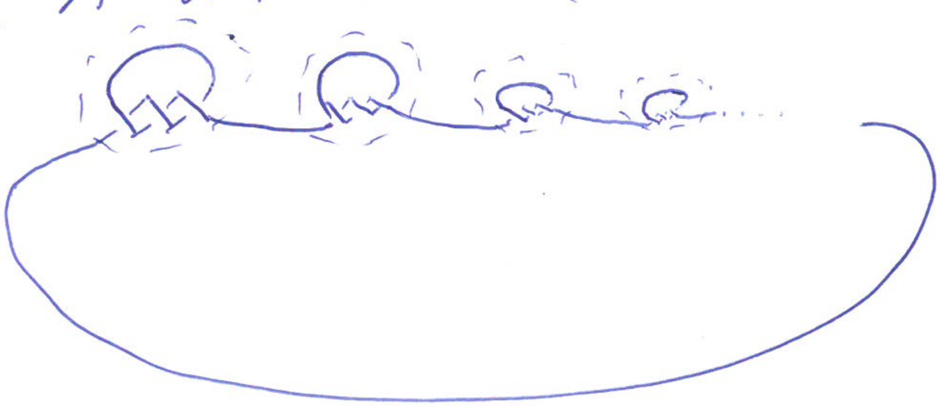
Bad definition of knot

A <u>knot</u> is the image of a one-to-one continuous map $f: S^1 \to \mathbb{R}^3$

* This gets us all "knots" from our intuitive def.

* Also gets us "wild" knots.

Ex| A wild knot (∞-knotting)



Bad def. of knot equivalence

Two knots $f: S^1 \to \mathbb{R}^3$ and $g: S^1 \to \mathbb{R}^3$ are equivalent if there is a continuous function

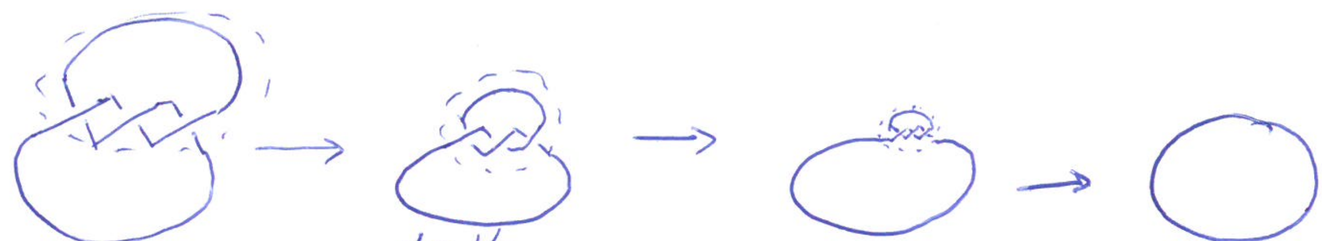
$$H: S^1 \times [0,1] \to \mathbb{R}^3 \quad \text{s.t. } \forall t \in [0,1]$$

$\cancel{H(x,t)}$ $H|_{S^1 \times \{t\}}$ is one-to-one,

$$H|_{S^1 \times \{0\}} = f \quad \text{and} \quad H|_{S^1 \times \{1\}} = g.$$

(we say f is isotopic to g).

Ex | All knots are equivalent.



$t=0$ trefoil

$t = 1/3$

$t = 2/3$

$t = 1$ "the unknot"

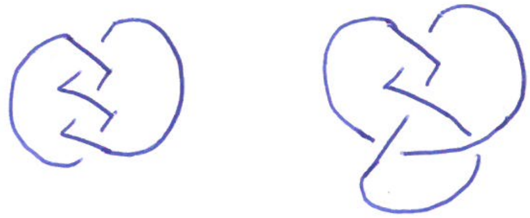
There are two solutions

1. Differential topology
   ~~A~~ Knots are smooth embeddings up to smooth ambient isotopies.

2. Piece-wise linear topology
   Knots are polygonal embeddings up to elementary deformations.

- Goal of knot theory is to tell knots apart.

Ex|



These are different.

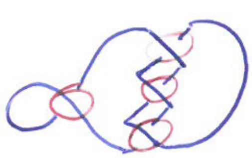
- The main tool for telling knots apart are knot invariants.

Def| A knot invariant is a function G from equivalence classes of knots $\mathcal{K}$ to some set of algebraic objects A.

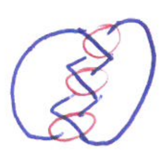
$$G : \mathcal{K} \to A.$$

Ex| The crossing number for a knot K is the minimal # of crossings in any diagram of K.

Ex|



Trefoil

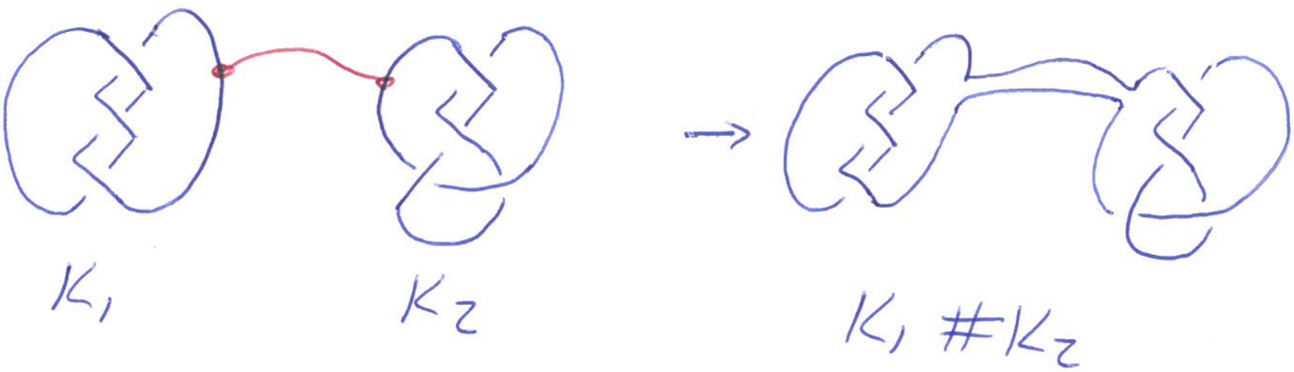


Trefoil

$c(\text{trefoil}) \leq 3$
(In fact $= 3$)

So, crossing number is a function from $\mathcal{K}$ to $\mathbb{Z}$.

Connected Sum



<u>Conj.</u> $c(K_1 \# K_2) = c(K_1) + c(K_2)$.
where $c(K)$ denotes the crossing number of K.

Grad. Knot Theory Day 2 f_n F6

Announcement

- HW 1 posted

Outline

- Def. of knot and knot equivalence
  - Differential topology
  - Piece-wise linear topology.

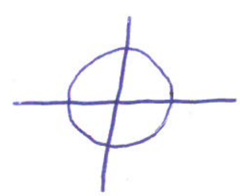
* Differential topology

<u>Def</u> A function $f: \mathbb{R} \to \mathbb{R}$ is <u>smooth</u> if $f^{(n)}: \mathbb{R} \to \mathbb{R}$ exist and are continuous for all n. (We say $f \in C^{\infty}(\mathbb{R})$ or f is ∞-ly differentiable.

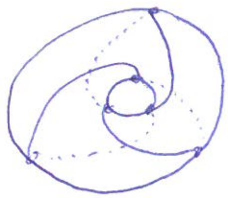
<u>Def</u> A function $f: \mathbb{R} \to \mathbb{R}$ is <u>smoothly periodic</u> of period ℓ if f is smooth, $f(x) = f(x+\ell)$ for all $x \in \mathbb{R}$ and $f^{(n)}(x) = f^{(n)}(x+\ell)$ for all n and for all $x \in \mathbb{R}$.

<u>Def</u> A <u>knot</u> is a parameterized curve $k: \mathbb{R} \to \mathbb{R}^3$ s.t. $k(t) = \langle x(t), y(t), z(t) \rangle$ and $x(t)$, $y(t)$ and $z(t)$ are all smoothly periodic of period 2π, and k is one-to-one.

<u>Ex</u> $k(t) = \langle \cos(t), \sin(t), 0 \rangle$



Ex 1 $k(t) = \langle (\cos(3t)+2)\cos(2t),$
$(\cos(3t)+2)\sin(2t),$
$-\sin(3t)\rangle$



A symetric embedding of the trefoil.

Question: Where is S^1?

Suppose $k:\mathbb{R}\to\mathbb{R}^3$ is a knot and $f:\mathbb{R}\to S^1$ s.t. $f(t)=e^{it}$, then there exists a unique map $K: S^1 \to \mathbb{R}^3$ s.t. the following diagram commutes.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "k"] \arrow[d, "f"'] & \mathbb{R}^3 \\ S^1 \arrow[ur, "K"'] & \end{tikzcd}$$

Note: For a more general space M. $H: M\to\mathbb{R}^n$ is <u>smooth</u> if H can be approximated by a linear map L and, when L is represented by an $n\times m$ matrix of functions, each of the $n\cdot m$ functions that appear as entries are all smooth.

Def Two knots $K_1 : S^1 \to \mathbb{R}^3$ and $K_2 : S^1 \to \mathbb{R}^3$ are <u>smoothly ambient isotopic</u> if there exists a smooth map $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ s.t.

1. for every $t \in [0,1]$
   - $H|_{\mathbb{R}^3 \times \{t\}}$ is smooth
   - $H|_{\mathbb{R}^3 \times \{t\}}$ is a bijection
   - $\left(H|_{\mathbb{R}^3 \times \{t\}}\right)^{-1}$ is smooth

   $\Rightarrow H|_{\mathbb{R}^3 \times \{t\}}$ is a diffeomorphism.

2. $\cancel{H|_{\mathbb{R}}}$ $H(K_1(t), 1) = K_2(t)$.

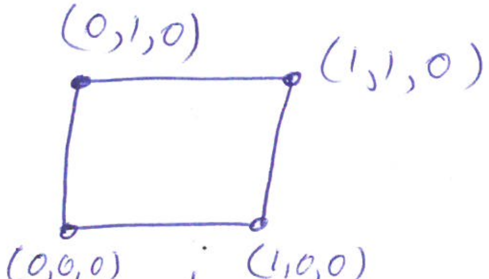
Note: Think of an ambient isotopy as a "fluid flow" on $\mathbb{R}^3$

* Peice-wise linear topology

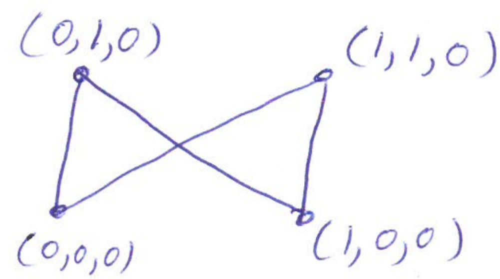
- Let $[p,q]$ denote the line segment between points $p, q \in \mathbb{R}^3$ in $\mathbb{R}^3$.
- Let $(p_1, \cdots, p_n)$ be an ordered set of distinct points in $\mathbb{R}^3$. $[p_1,p_2] \cup [p_2,p_3] \cup \cdots \cup [p_{n-1},p_n] \cup [p_n,p_1]$ is a simple closed ~~curve~~ polygonal curve if each segment in $\{[p_1,p_2], \cdots, [p_n,p_1]\}$ ~~mea~~ intersects the union of the remaining segment only in its end points.

Def A knot is a simple closed polygonal curve in $\mathbb{R}^3$.

Ex



knot



not a knot.

Def Given knots J and K, J is an elementary deformation of K if one of J and K is $(P_1, P_2, \cdots, P_n)$ and the other is $(P_0, P_1, \cdots, P_n)$ s.t.

① P_0 is not collinear with P_1 and P_n

② The convex hull of (P_0, P_1, P_n) intersects $[P_1,P_2] \cup \cdots \cup [P_n,P_1]$ only in $[P_1,P_n]$

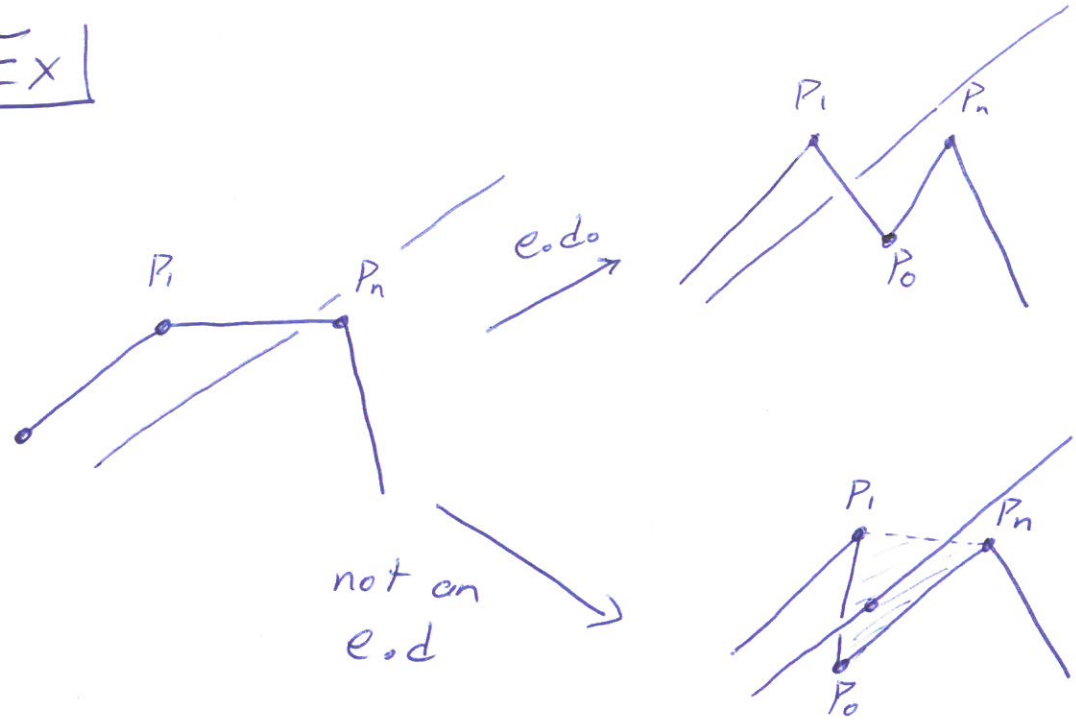
H.W. (1) Uniqueness and existance condition.

(2) Quardrilaterals are unknots

(3) Define torus knot, ask them to find eq. of torus.

(4) Ask them to prove the wiggle ~~lemma~~ lemma. for Poly.

Ex



Def Knots K and J are equivalent if there exists a sequence of nots $K = K_0, K_1, \ldots, K_n = J$ s.t. K_{i+1} is an elementary deformation of K_i for $0 \le i \le n-1$.

* These seem different, however, the Piece-wise linear theory of knots and the differentiable theory of knots are equivalent.

Grad Knot Theory Day 3

Outline
- Knots invariant under scaling via smooth definitions
- Knots invariant under translation via P.L. definitions

- Last time we saw

<u>Def</u> A knot $k: \mathbb{R} \to \mathbb{R}^3$ is a 1-1 parameterization s.t. $k(t) = \langle x(t), y(t), z(t) \rangle$ and each of $x(t)$, $y(t)$ and $z(t)$ are smoothly periodic of period 2π

<u>Def</u> Two knots $K_1: \mathbb{R} \to \mathbb{R}^3$ and $K_2: \mathbb{R} \to \mathbb{R}^3$ are <u>smoothly</u> <u>abient</u> <u>isotopic</u> if there exists a smooth function $H: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$

s.t. ① for every $t \in [0,1]$, $H|_{\mathbb{R}^3 \times \{t\}}$ is a smooth bijection with smooth inverse

② $H(K_1(t), 1) = K_2(t)$

③ $H(\vec{x}, 0) = \vec{x}$

Claim Given a knot $k: \mathbb{R} \to \mathbb{R}^3$ s.t.
$k(t) = \langle x(t), y(t), z(t) \rangle$ Let λk denote the knot $\lambda k(t) = \langle \lambda x(t), \lambda y(t), \lambda z(t) \rangle$ for $\lambda > 0$.
Then k is smoothly ambient isotopic to λk.

Pf Define $H: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$
s.t. $H(x,y,z,t) = \big(((\lambda-1)t+1)x,$
$((\lambda-1)t+1)y,$
$((\lambda-1)t+1)z \big).$

Fact: Since the component functions of H are polynomials, then H is smooth.

① Examine $H(x,y,z,0) = (x,y,z)$ ✓

② Examine $H(k(t),1) = H(x(t), y(t), z(t), 1)$
$= \langle \lambda x(t), \lambda y(t), \lambda z(t) \rangle$
$= \lambda k(t)$ ✓

③ Fix $t \in [0,1]$

$$H\big|_{\mathbb{R}^3 \times \{t\}}(x,y,z) = \begin{bmatrix} (\lambda-1)t+1 & 0 & 0 \\ 0 & (\lambda-1)t+1 & 0 \\ 0 & 0 & (\lambda-1)t+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $\operatorname{Ker}(M) = \{0\}$, so $H\big|_{\mathbb{R}^3 \times \{t\}}$ is a bijection.
- Since all linear maps are smooth, $H\big|_{\mathbb{R}^3 \times \{t\}}$ and $\left(H\big|_{\mathbb{R}^3 \times \{t\}}\right)^{-1}$ are smooth.

Thus, k is smoothly ambient isotopic to λk. $\square$

- Last time

Def | A knot is a simple closed polygonal curve.

Def | Two such knots are equivalent if they are related by a finite sequence of elementary deformations.

Def | $Y \subset \mathbb{R}^n$ is closed if it contains all of its limit points.

Def | $Y \subset \mathbb{R}^n$ is bounded if there exists $N \in \mathbb{R}_{>0}$ s.t. $Y \subset B_N(\vec{0})$.

Def | $Y \subset \mathbb{R}^n$ is compact if Y is closed and bounded

Def | If $Y \subset \mathbb{R}^n$ and $\varepsilon > 0$, then $\eta_\varepsilon(Y)$ denotes the set of all points in $\mathbb{R}^n$ ~~s.t.~~ that are within distance ε of Y.

Thm | If X and Y are disjiont compact subsets of $\mathbb{R}^n$, then $\exists\, \varepsilon > 0$ s.t.

$\eta_\varepsilon(X) \cap \eta_\varepsilon(Y) = \emptyset$.

Proof | Real analysis.

<u>Lemma</u> Let K be a knot determined by the ordered set of points $(P_1, \ldots, P_n)$. There exists $\varepsilon > 0$ s.t. if $d(P_1, P_1') < \varepsilon$, then K is equivalent to the knot determined by $(P_1', P_2, \ldots, P_n)$.

<u>Pf</u> Since $[P_1, P_2]$ and $[P_3, P_4] \cup \cdots \cup [P_{n-1}, P_n]$ are compact ~~them~~ and disjiont, then $\exists \varepsilon_1 > 0$ s.t. $\eta_{\varepsilon_1}([P_1, P_2]) \cap \eta_{\varepsilon_1}([P_3, P_4] \cup \cdots \cup [P_{n-1}, P_n]) = \emptyset$.

Similarly, $\exists \varepsilon_2 > 0$ s.t.

$$\eta_{\varepsilon_2}([P_n, P_1]) \cap \eta_{\varepsilon_2}([P_2, P_3] \cup \cdots \cup [P_{n-2}, P_{n-1}]) = \emptyset.$$

~~Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.~~

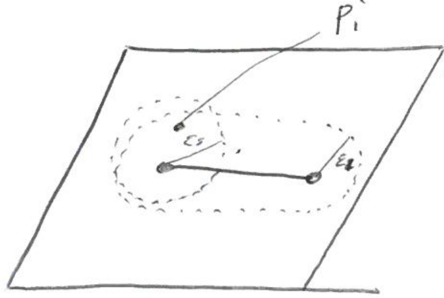
~~Suppose $P_1' \in \mathbb{R}^3$ s.t. $d(P_1, P_1') < \varepsilon$ and P_1' is not colinear with (P_1, P_2) and not colinear with (P_n, P_1).~~

Since $\{P_1\}$ and $[P_2, P_3] \cup \cdots [P_{n-1}, P_n]$ are compact and disjiont, then $\exists\ \varepsilon_3 > 0$ s.t.

$$\eta_{\varepsilon_3}(\{P_1\}) \cap \eta_{\varepsilon_3}([P_2, P_3] \cup \cdots \cup [P_{n-1}, P_n]) = \emptyset.$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Suppose $P_1' \in \mathbb{R}^3$ s.t. $d(P_1, P_1') < \varepsilon$ and P_1' is not colinear with P_1, P_2 and not colinear with P_n, P_1. WLG assume $d(P_1', [P_1, P_2]) \leq d(P_1', [P_n, P_1])$. $**$



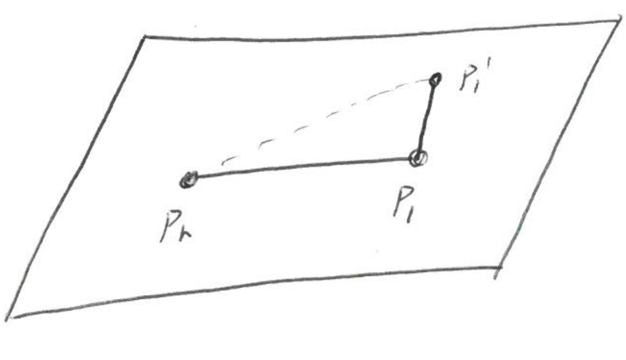
$\leftarrow$ plane L containing P_1', P_1 and P_2

Since the triangle spaned by P_1' P_1 and P_2 is contained in $\eta_{\varepsilon_1}([P_1, P_2])$, then it is disjiont from $[P_3, P_4] \cup \cdots \cup [P_{n-1}, P_n]$.

If the triangle spanned by $P_1' P_1$ and P_2 intersects $[P_2, P_3]$, then $P_3 \in L$ and $P_3 \notin \eta_{\varepsilon_1}([P_1, P_2])$. This implies the line segment $[P_2, P_3]$ intersects $\eta_{\varepsilon_3}(P_1)$, a contradiction.

By a similar argument $[P_n, P_1]$ is disjiont from the triangle spaned by P_1', P_1 and P_2.

Hence $(P_1, P_1', P_2, \ldots, P_n)$ is equivalent to $(P_1, P_2, \ldots, P_n)$.



plane P containing P_1', P_1, P_n

Since $P_1' \in \eta_{\varepsilon_2}([P_n, P_1])$ then

The $\Delta P_1' P_n P_1$ is disjiont from $[P_2, P_3] \cup \cdots \cup [P_{n-2}, P_n]$.

As before, if $[P_{n-1}, P_n)$ or $(P_1', P_n]$ intersect we obtain contradictions. So,

$(P_1, P_1', P_2, \ldots, P_n)$ is equivalent to

$(P_1', P_2, \ldots, P_n)$.

The case when P_1' is collinear to P_1, P_2 or to P_n, P_1 can be handled similarly. $\square$

Grad Knot Theory Day 4

Announcements

- HW 1 due Tues. at ~~mig~~ midnight

Last time

Lemma Given a knot K defined by $(p_1, p_2, \ldots, p_n)$, there exists $\varepsilon > 0$ s.t. if $\|p_1' - p_1\| < \varepsilon$, then K is equivalent to the knot defined by $(p_1', p_2, \ldots, p_n)$.

Pf $\exists\, \varepsilon_1 > 0$ s.t. $\eta_{\varepsilon_1}(\{P_1\}) \cap \eta_{\varepsilon_1}(\bigcup_{i=2}^{n-1} [P_i, P_{i+1}]) = \emptyset$

$\varepsilon_2 > 0 \quad \cdots \quad [P_1, P_2] \quad \bigcup_{i=3}^{n-1} [P_i, P_{i+1}]$

$\varepsilon_3 > 0 \quad \cdots \quad [P_n, P_1] \quad \bigcup_{i=2}^{n-2} [P_i, P_{i+1}]$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and let $\|P_1' - P_1\| < \varepsilon$

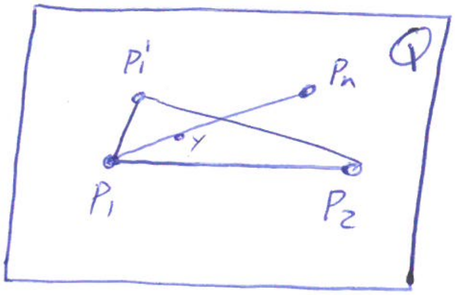
More over if P_1, P_2, P_n and P_1' are coplanar relable s.t. $\measuredangle P_1' P_1 P_2 < \measuredangle P_1' P_1 P_n$.

Case 1: P_1' is <u>not</u> colinear with P_1, P_2 <u>and</u> P_1' is <u>not</u> colinear with P_1, P_n.

Case 1a: If Q is the plane containing P_1', P_1 and P_2, then ~~both~~ neither P_n nor P_3 are contained in Q. Last time we showed there is an elementary deformation taking $(P_1, P_2, \ldots, P_n)$ to $(P_1, P_1', P_2, \ldots, P_n)$.

Case 1b: Assume toward a $\divideontimes$ that

$[P_1, P_n] \cap \Delta P_1 P_1' P_2 \ni y$ s.t. $y \neq P_1$



Then $P_n \in Q$.

Moreover, we can conclude $\measuredangle P_1' P_1 P_n < \measuredangle P_1' P_1 P_2$ or P_1' is colinear with P_1, P_n. However both assumptions $\divideontimes$ our hypothesis, so

$[P_1, P_n] \cap \Delta P_1 P_1' P_2 = \{P_1\}$

Assume toward a $\divideontimes$ that

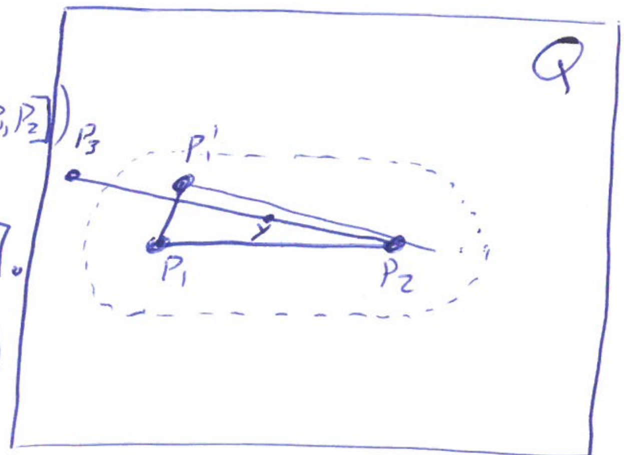
$[P_2, P_3] \cap \Delta P_1 P_1' P_2 \ni y$ s.t. $y \neq P_2$

Then $P_2 \in Q$.

By def. of ε_2, $P_3 \notin \eta_{\varepsilon_2}([P_1, P_2])$

Hence, $\exists w \in [P_1, P_1'] \cap [P_2, P_3]$.

However, $[P_1', P_1] \subset \eta_{\varepsilon_1}(\{P_1\})$



and $\eta_{\varepsilon_1}(\{P_1\})$ is disjiont from $[P_2, P_3]$, a contradiction. Hence, $[P_2, P_3] \cap \Delta P_1 P_1' P_2 = \{P_2\}$.

Finally, since $\Delta P_1 P_1' P_2 \subset \eta_{\varepsilon_2}([P_1, P_2])$, $[P_3, P_4] \cup \cdots \cup [P_{n-1}, P_n]$ is disjiont from $\Delta P_1 P_1' P_2$.

~~These~~ Thus, $\Delta P_1' P_1 P_2$ meets K in exactly $[P_1, P_2]$ and K is equivalent to $(P_1, P_1', P_2, \ldots, P_n)$.

left to show • $(P_1, P_1', P_2, \dots, P_n)$ is equivalent to $(P_1', P_2, \dots, P_n)$

• Deal with the cases when P_1' is colinear to P_1, P_2 or P_1, P_n.

Knot Diagrams

Let $P : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map

$P(x, y, z) = (x, y)$

<u>Def</u> A ~~knot~~ <u>projection</u> of a knot K is the image of $P|_K$ (or $P(k(t))$ for k a smooth knot)

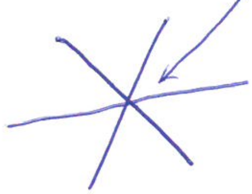
<u>Ex</u>

a projection of the fig. 8 knot.

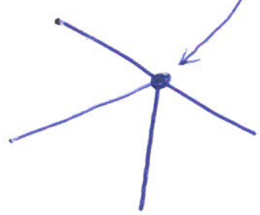
<u>Def</u> A projection of a polygonal knot is <u>regular</u> if no three points on the knot project to the same point on the projection and if two points project to the same point, neither is a vertex.

<u>Ex</u>

Bad

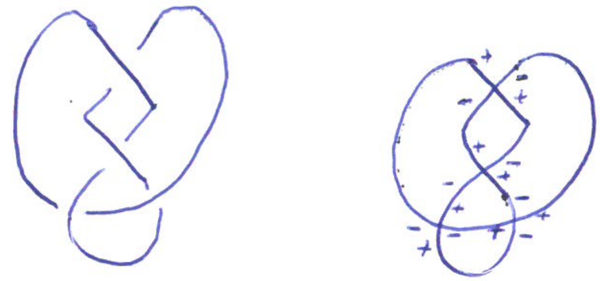


Bad



Def| A knot diagram is a regular knot projection together with lables that indicate which strand goes over at each double point.

Ex|



Generalization of our previous lemma|

If K is a polygonal knot determined by $(P_1, P_2, \dots, P_n)$ $\exists\, \varepsilon > 0$ s.t. if $\|P_i' - P_i\| < \varepsilon$ for all $1 \le i \le n$, then K is equivalent to the knot determined by $(P_1', P_2', \dots, P_n')$.

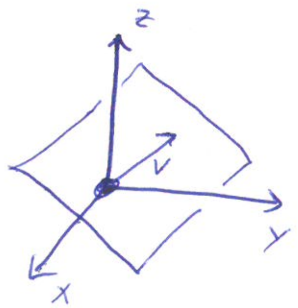
Th^m| Let K be a knot determined by $(P_1, \dots, P_n)$. $\forall t > 0$ $\exists$ a knot K' determined by $(q_1, \dots, q_n)$ s.t. the distance from q_i to P_i is less than t for all i and K' has a regular projection.

Cor| Every knot is equivalent to a knot with a regular projection.

Pf of Th^m| One can show that if $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an orientation preserving rigid motion (i.e. L is a translation or rotation) then K is equivalent to $L(K)$.

Let $v \in S^2$

Let $P_v : \mathbb{R}^3 \to \mathbb{R}^2$ be the orthoganal projection onto the plane in $\mathbb{R}^3$ that is perendicular to v.



If K is a polygonal knot in $\mathbb{R}^3$, one can show that

① The set of all $v \in S^2$ s.t. three points in K get mapped to a single point by P_v is ~~finite~~ contained in a finite union of great circles.

② The set of all $v \in S^2$ s.t. a vertex of K and a distinct point of K get mapped to the same point by P_v is contained in a finite union of great circles on S^2.

Hence, $P_v(K)$ is a regular projection for all but a set of measure zero.

Grad knot theory Day 5

Announcement

- HW 1 Due today.

Last time

- Knot projection (image of $p|_K$)
- regular knot projection

no  and no 

- Knot diagram

- **Thm** Every polygonal knot is equivalent to a knot with a regular projection.

Thm If J and K are polygonal knots with identical diagrams, then J is equivalent to K.

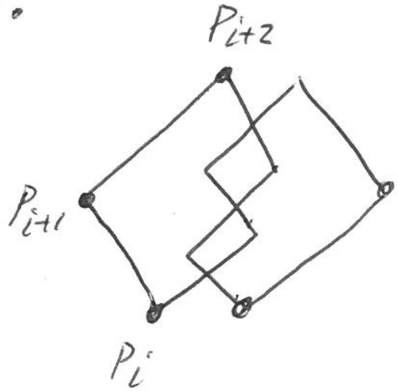
Sketch of proof

Suppose K is defined by $(p_1, p_2, \dots, p_n)$ and J is defined by $(q_1, q_2, \dots, q_m)$.
After adding vertices to K and J, we can assume,

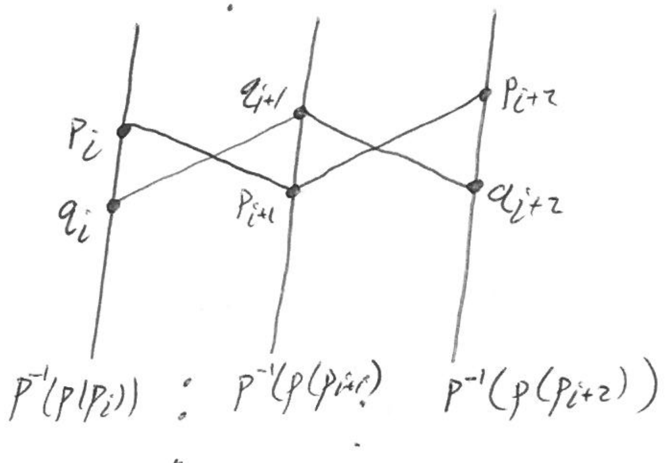
$n = m$ and (if $p(x,y,z) = (x,y)$) $p(P_i) = p(q_i)$ for all $1 \le i \le n$.

<u>Case 1</u>) Suppose $p([P_i, P_{i+1}] \cup [P_{i+1}, P_{i+2}])$ is disjiont from any double point of $p(K) = p(J)$.

<u>Ex</u>)



Examine $(K \cup J) \cap p^{-1}(p([P_i, P_{i+1}] \cup [P_{i+1}, P_{i+2}]))$



Since $p([P_i, P_{i+1}])$ is disjion from any double point of $p(K)$, then $p^{-1}(p([P_i, P_{i+1}])) \cap K = [P_i, P_{i+1}]$.
Moreover, the triangle $\Delta P_i, P_{i+1}, q_{i+1}$ defines an elementary deformation showing $(P_1, P_2, \dots, P_n)$ is equivalent to
$(P_1, \dots, P_i, q_{i+1}, P_{i+1}, \dots, P_n)$.

Similarly, $(P_1, P_2, \ldots, P_i, q_{i+1}, P_{i+1}, \ldots, P_n)$
is equivalent to $(P_1, P_2, \ldots, P_i, q_{i+1}, P_{i+2}, \ldots, P_n)$.

Case 2: Suppose $p([P_i, P_{i+1}] \cup [P_{i+1}, P_{i+2}])$ meets a double point of $p(K)$. and Show $(P_1, \ldots, P_n)$ is equivalent to

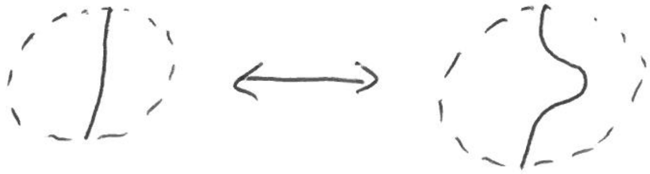
$(P_1, \ldots, P_i, q_{i+1}, P_{i+2}, \ldots, P_n)$.

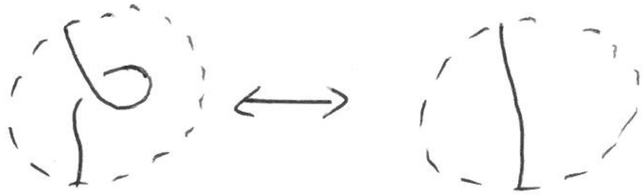
Pf] Exercise

Procede by induction to show $(P_1, \ldots, P_n)$ is equivalent to $(q_1, \ldots, q_n)$.

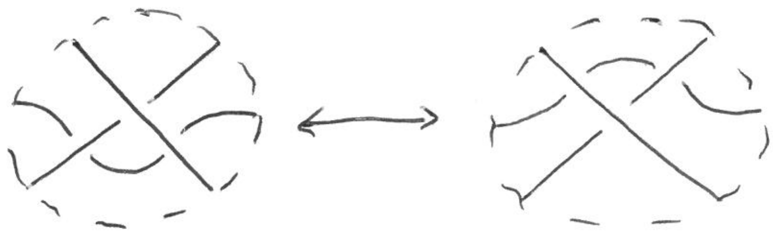
Thm (Alexander–Briggs, Reidermeister)
Two knots are equivalent iff their diagrams are related via a finite sequence of Reidermeister moves.

Reidermeister moves

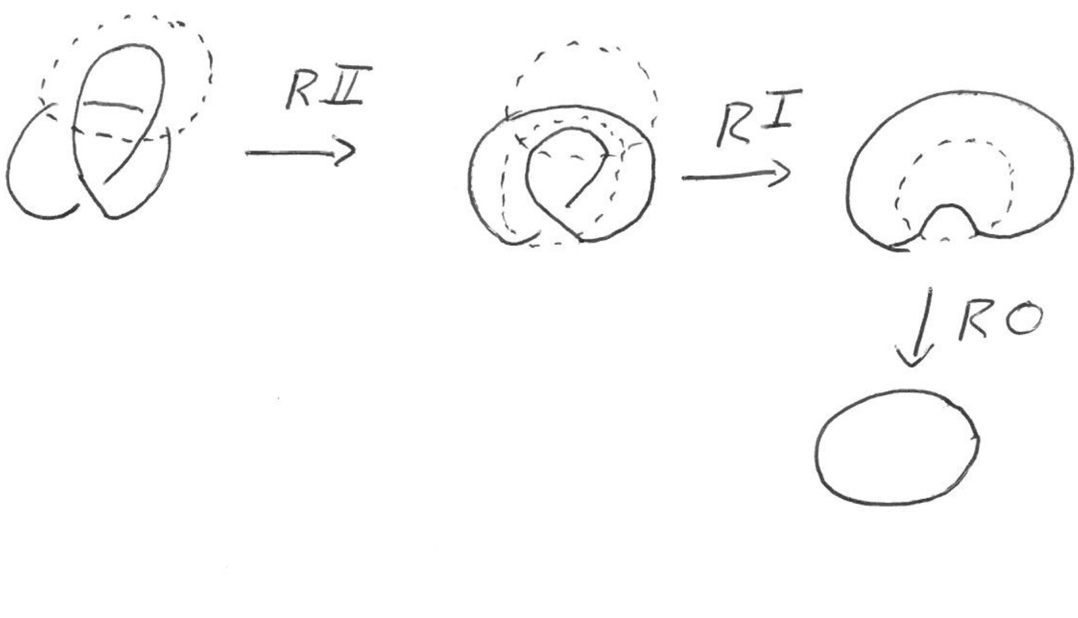
R0:  (planar isotopy)

RI: 

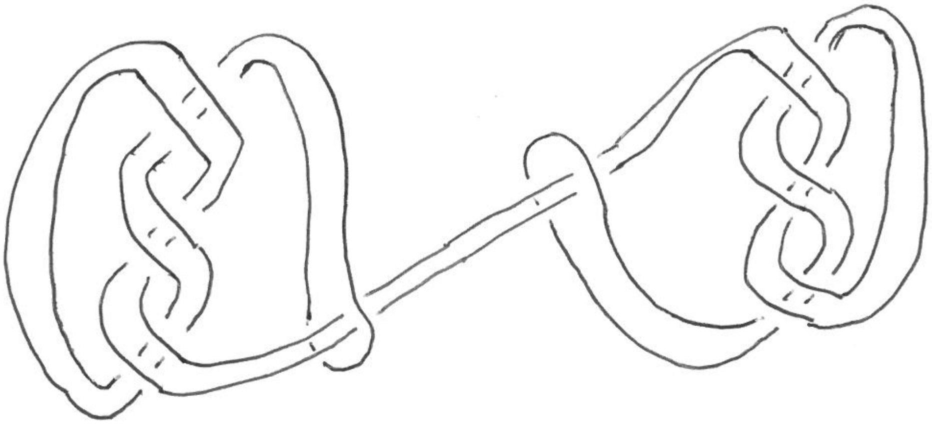
RII: 

RIII: 

<u>Ex 1</u>



Monster Hard unknots



"Cousin it"

<u>Th^{m}</u> (Hass and Lagaris)
Any unknotted knot diagram with n
crossings can be transformed to the
trivial knot diagram using at most
$2^{(10^{11})n}$ Reidermeister moves.

Grad Knot Theory day 6

Knot Invariants

Thm (Alexander - Briggs, Reidemeister)
Two knots are equivalent iff their diagrams are related via a finite sequence of Reidemeister moves.

Colorability of knots (Due to Ralph Fox)

B = blue
R = red
G = green

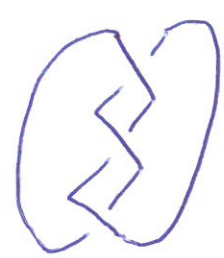
Def A knot is colorable 3-colorable if it has a diagram D s.t. each arc of D is colored Blue, Red or Green, at least two colors are used and at any crossing where two colors appear, all three appear.

Crossing criteria. → at any crossing where two colors appear, all three appear.

Ex

not colorable

not colorable



colorable.

Recall that a <u>knot invariant</u> is a function $f: \mathcal{K} \to A$ where $\mathcal{K}$ is the set of equivalence classes of knots and A is some set of algebraic structures.

Define $g: \mathcal{K} \to \{0, 1\}$ where

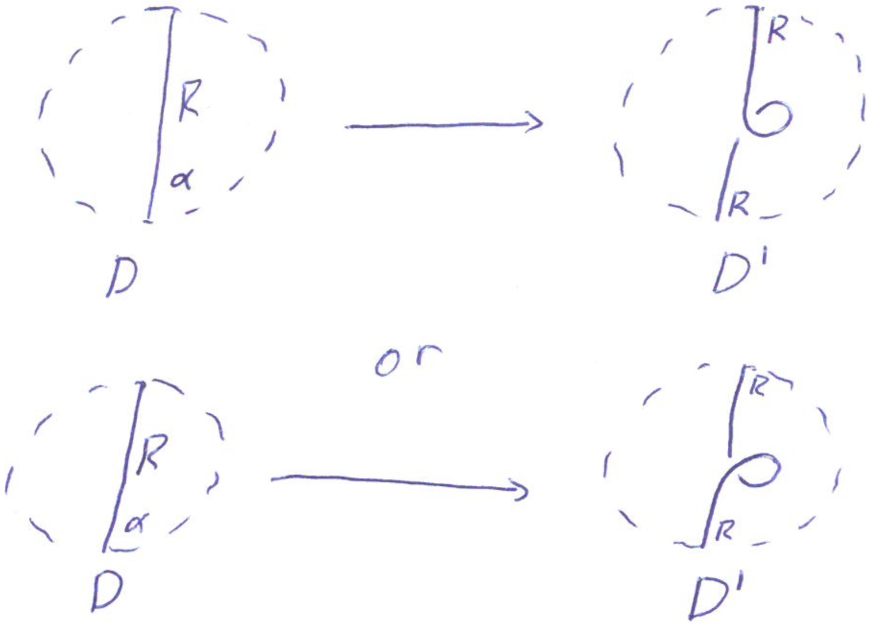
$g([K]) = 1$ if K has a colorable diagram

$g([K]) = 0$ if K has a diagram that is not colorable.

By Alexander-Briggs, to show g is an invariant we must show that colorability is preserved under each Reidemeister move.

<u>Pf</u> Claim 1: Colorability is preserved under R0. Clearly True

Claim 2: Colorability is preserved under RI. Let D be a colorable diagram and D' be the result of on RI move on D' Suppose D' has one more crossing than D and α, the arc of D involved in the RI move recieves the color red.



Since D is colorable, then at least two distinct colors are used to color the arcs of D.
Since D' is identical to D outside of domain of Redemeister move, then at least two colors are used to color the strands of D'.

The crossing criteria holds at all crossings ~~out side~~ of D' outside of the domain of the ~~reider~~ Reidemeister move, since D is identical to D' in that region. As depicted in the fig. above, the crossing criteria holds at the single new crossing of D'.
Thus D' is colorable

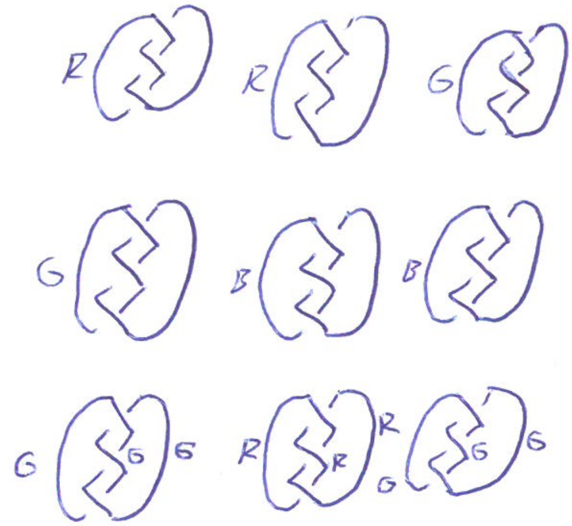
Claim 3: Exercise: Colorability is preserved under R II

Claim 4: Exercise: Colorability is presered under R III.

Moreover, the number of 3-colorings of a knot is a knot invariant. $c_3(K)$

Ex|

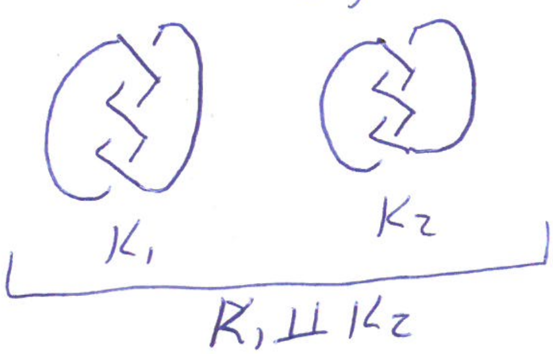
$C_3(\text{unknot}) = 3$



$C_3(\text{trefoil}) = 9$

Ex| $C_3(K_1 \sqcup K_2) = C_3(K_1) \cdot C_3(K_2)$

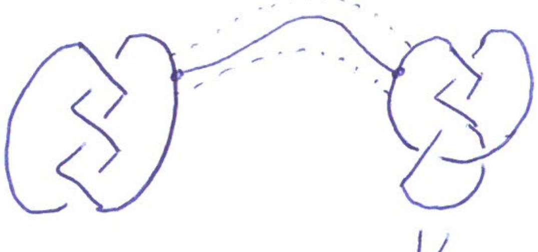
disjiont union



Pick a coloring of K_1 and a coloring of K_2, this induces a coloring of $K_1 \sqcup K_2$.
So, $C_3(K_1 \sqcup K_2) \geq C_3(K_1) \cdot C_3(K_2)$
Moreover, if you have a coloring of $K_1 \sqcup K_2$, this induces a coloring on K_1 and a coloring on K_2,
so, $C_3(K_1 \sqcup K_2) \leq C_3(K_1) \cdot C_3(K_2)$

Recall: Connected sum

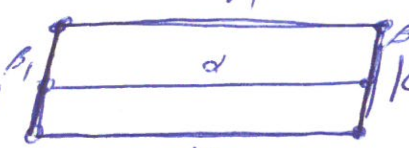
Given knots K_1 and K_2, there exists a regular projection for $K_1 \amalg K_2$ s.t. $p(K_1) \cap p(K_2) = \emptyset$.



K_1 K_2

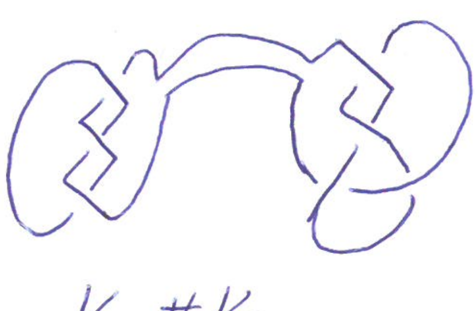
Let α be a closed arc in the xy-plane s.t. $\partial(\alpha) = \{x_1, x_2\}$, $\alpha \cap p(K_1) = \{x_1\}$, $\alpha \cap p(K_2) = \{x_2\}$

There is a rectangular nbh of α of the following form:



$K_1 \# K_2$ is the knot with the diagram

$$\left(p(K_1 \amalg K_2) \setminus (\beta_1 \cup \beta_2)\right) \cup (\gamma_1 \cup \gamma_2).$$



$K_1 \# K_2$

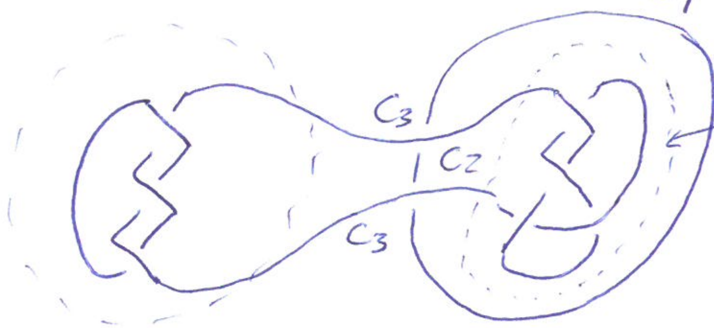
Key fact: The equivalence classes of K_1 and K_2 determine the equivalence class of $K_1 \# K_2$.

Ex $C_3(K_1 \# K_2) = \frac{1}{3} C_3(K_1) \cdot C_3(K_2)$

Pf $C_3(U \amalg K_1 \# K_2) = 3 \cdot C_3(K_1 \# K_2)$

c_1 (3 choices)

$\frac{1}{3} C_3(K_1)$ choices



$\frac{1}{3} C_3(K_2)$ choices

c_2 (3 choices)

$$3 \cdot C_3(K_1 \# K_2) = 3 \cdot 3 \cdot \tfrac{1}{3} \cdot \tfrac{1}{3} \cdot C_3(K_1) \cdot C_3(K_2)$$

$$\boxed{C_3(K_1 \# K_2) = \tfrac{1}{3} C_3(K_1) \cdot C_3(K_2)}$$

Grad knot theory Day 7

Last time|

- 3-colorability
- # of 3-coloings ($C_3(K)$)

$\Big\}$ Knot Invariants.

Outline|

- Unknotting number

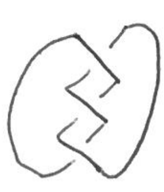
Def| The unknotting number of a knot K is the minimal number of crossing changes nessisary over all diagrams of K to change K to the unknot. (denoted $u(K)$)

Ex|

So $u(7_2) = 1$

Def| The crossing number of K is the minimal number of crossings in any diagram of K. (denoted $c(K)$).

Ex|



$c(\text{trefoil}) = 3$

Exercise| Prove $c(\text{trefoil}) = 3$

Def Let $h: \mathbb{R}^3 \to \mathbb{R}$ s.t. $h(x,y,z) = z$.

Let K be a polygonal knot in $\mathbb{R}^3$. K is equivalent to a knot s.t. no edge of K is parallel to the xy-plane.

Hence, all vertices of K are:

local maxima   local minima   or neither

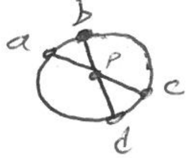
The <u>bridge number</u> of K is the minimal number of local maxima over all knots equivalent to K and have no edges parallel to the xy-plane. (denoted $\beta(K)$).

Thm If $\beta(K) = 1$, then K is an unknot

Idea of proof By Rolle's thm every plane parallel to the xy-plane meets K in 0, 1 or 2 points. This can be used to show that the top 3 vertices span a triangle that defines an elementary deformation that reduces the number of vertices by 1, ~~but preserves the~~ and results in a knot of bridge number 1. By induction, K is the unknot.

Claim If $c(K)=1$, the K is the unknot.

Let p be the single double point of $P|_K$.
Since $P|_K$ is a regular projection, $\exists\ \varepsilon>0$ s.t. $P|_K$ meets the closed ball of radius ε centered at p, $\overline{B}_\varepsilon(p)$, in an "x". i.e.

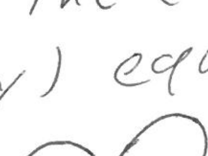


Let $\{a,b,c,d\} = \partial\overline{B}_\varepsilon(p) \cap P|_K(K)$

Since $(\overline{B}_\varepsilon(p))^c$ is incident to no double points of $P|_K$, then $\text{Image}(P|_K) \cap (\overline{B}_\varepsilon(p))^c = \alpha \cup \beta$ where α and β are two planar arcs.

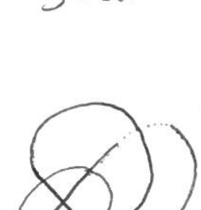
Case 1: α connects b to a and β connects c to d.
Case 2: α connects a to d and β connects b to c.
Case 3: α connects b to d and β connects a to c

In Cases 1 and 2, the diagram of K is (up to planar isotopy) equivalent to

 or 

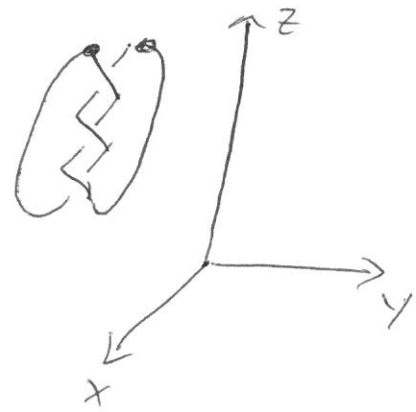
Applying a type I Reidemeister move to this diagram produces the standard diagram of the unknot.

Case 3: is impossible



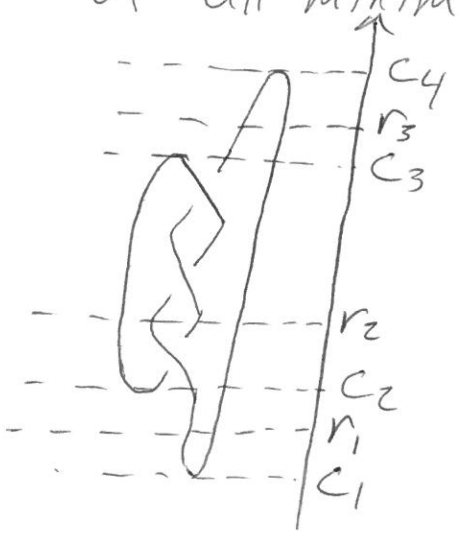
Since any two simple closed polygonal curves in the plane with a regular projection intersect in an even # of points. $\square$

Ex| $\beta(\text{trefoil}) = 2$



Width| Assume K has be modified s.t. no edge of K is parallel to the xy-plane and no two vertices of K ~~occur at~~ are mapped to the same value by h.

Let $c_1 < c_2 < \cdots < c_{n+1}$ be the images of all minima and maxima under h.



~~Let~~ Choose $r_i \in \mathbb{R}$ s.t. $c_i < r_i < c_{i+1}$ for $0 \le i \le n$.

Define the complexity

$$\sum_{i=1}^{n} |K \cap h^{-1}(r_i)|.$$

Def| The <u>width</u> of K is the minimum value of this complexity over all suitable knots equivalent to K. (denoted $w(K)$).

Theorems and open questions

Conj $u(K_1 \# K_2) = u(K_1) + u(K_2)$

Thm (Scharlemann 85)
If $u(K_1 \# K_2) = 1$, then K_1 or K_2 is the unknot.
Pf Very Hard

Conj $c(K_1 \# K_2) = c(K_1) + c(K_2)$

Thm (Kauffman, Murasugi, Thistlethwaite 88)
If K_1 and K_2 are alternating knots, then
$c(K_1 \# K_2) = c(K_1) + c(K_2)$
Pf we will do it!

Thm (Lackenby, 09)
$\frac{1}{152}(c(K_1) + c(K_2)) \leq c(K_1 \# K_2) \leq c(K_1) + c(K_2)$

Thm (Schubert 54)
$\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1$

Conj $w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$

Thm (Rieck & Sedgewick) If K_1 and K_2 are ~~meridionaly sma~~ mp-small, then $w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$.

Thm (Blair & Tomova) $\exists\ K_1, K_2$ s.t.
$w(K_1 \# K_2) = \max(w(K_1), w(K_2)) < w(K_1) + w(K_2) - 2$.