MATH 555: KNOT THEORY, HOMEWORK 2

REIDEMEISTER MOVES, COLORABILITY, KNOT INVARIANTS

Due by Firday, Feb. 8st at 10 am

1. Homework Policy

You are strongly encouraged to work in groups to exchange ideas and help each other understand how to approach problems, but the work you turn in must be your own! If you work with others on an assignment, be sure to indicate the names of the other students on your homework. Additionally, if you use any outside resources (i.e. internet sources, other mathematicians, other books) to help you solve homework problems, you must cite your sources. Failure to follow these rules will result in a score of zero on an assignment and may constitute a violation of academic integrity.

Homework must be legible, well-organized, and written in complete sentences. Handwritten work is fine, but you are encouraged to type up the problems in LaTeX.

2. Readings and Responses.

(1) Read Sections 3.1, 3.2 and 3.3 of "Knots Knotes" by Justin Roberts.

3. Problems

(1) Let $P \subset \mathbb{R}^2$ be a regular knot projection. Describe how to construct an unknot with projection P. (Hint: use the fact that the only knot with bridge number equal to one is the unknot.)

(2) Using the proof for R1 presented in class as a model, carefully show that 3-colorability is preserved under the R2 Reidemeister move. (Hint: a clever organization of the cases of the proof will save you time.)

(3) Do Exercise 3.3.10 in "Knots Knotes".

(4) Let $\mathbb{Z}_p$ denote the set of integers mod p where p is a prime number. Recall that $\mathbb{Z}_p$ forms and abelian group under addition mod p. Moreover, $\mathbb{Z}_p$ is a field with multiplication defined to be multiplication mod p.

A p-coloring of a knot diagram D, where p is an odd prime, is a labeling of the arcs of D by elements of $\mathbb{Z}_p$ so that at every crossing

$$2x - y - z = 0 \; mod \; p$$

where x is the label of the over arc, y is the label of one under arc and z is the label of the other under arc. (Note that labeling all the arcs the same value is always a valid p-coloring under this definition.)

A) Given a knot diagram D, show that the number of distinct p-colorings of D is a power of p. (Hint: Show that set of p-colorings is in one-to-one correspondence to the kernel of a linear map from V to W where V and W are finite dimensional vectorspaces over the field $\mathbb{Z}_p$.)

B) Using only four illustrations and only four complete sentences, convince me that the number of p-colorings of a knot diagram is a knot invariant.

C) Find all odd prime numbers p such that the figure eight knot has a non-trivial p-coloring (i.e. one that involves at least two distinct labels). (Hint: Think about how Gaussian elimination works when your entries are elements in the field $\mathbb{Z}_p$)