Lec. 8

Th^m (Preimage Th^m)

If y is a regular value of $f: X \to Y$, then the preimage $f^{-1}(y)$ is a submanifold of X with $\dim(f^{-1}(y)) = \dim X - \dim Y$.

Recall: (Def) Given a smooth map $f: X \to Y$, $y \in Y$ is a <u>regular point</u> if for every $x \in f^{-1}(y)$ f is a submersion at x.

pf of th^m Note $f^{-1}(y) \subset X \subset \mathbb{R}^N$ for some N.

Let $x \in f^{-1}(y)$. Since y is a regular point, f is a submersion at x.

By the Local submersion theorem, there exist local parameterizations ϕ and γ which give rise to local coordinates about x and $f(x) = y$ s.t.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"'] & V \arrow[u, "\gamma"'] \end{tikzcd}$$

$\phi(0) = x$
$\gamma(0) = y$

and $f(x_1, \dots, x_k) = (x_1, \dots, x_l)$

Since y has the value $(0, 0, \dots, 0)$ in local coordinates, then $f^{-1}(y)$ is given by $\{(x_1, \dots, x_k) \in \phi(\mathcal{U}) \mid x_1 = \dots = x_\ell = 0\}$.

Let $\mathcal{U}'$ be the open subset of $\mathbb{R}^{k-\ell}$ given by $\mathcal{U} \cap \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1 = \dots = x_\ell = 0\}$.

Then $\phi|_{\mathcal{U}'} : \mathcal{U}' \longrightarrow \phi(\mathcal{U}) \cap f^{-1}(y) = \phi(\mathcal{U}')$ is a diffeomorphism.

Hence, $f^{-1}(y)$ is a submanifold of X of dimension $\dim(X) - \dim(Y)$. $\square$

$\boxed{Ex}$ Let $M_2(\mathbb{R})$ be the set of all 2×2, real-valued matrices.

$$M_2(\mathbb{R}) = \mathbb{R}^4$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [a \;\; b \;\; c \;\; d]$$

$$SL_2(\mathbb{R}) = \{ A \in M_2(\mathbb{R}) = \mathbb{R}^4 \mid \det(A) = 1 \}$$

Let $f : \mathbb{R}^4 = M_2(\mathbb{R}) \to \mathbb{R}$

by $f(x_1, x_2, x_3, x_4) = x_1 \cdot x_4 - x_2 \cdot x_3$

$$df_x = \underset{4\times 4}{[x_4 \;\; -x_3 \;\; -x_2 \;\; x_4]}$$

So df_x has full rank as long as $x \neq (0,0,0,0)$.

Hence $y \in \mathbb{R}$ s.t. $y \neq 0$ is a regular value of f.

So, $f^{-1}(1) = SL_2(\mathbb{R})$ is a 3-dim'l submanifold of $\mathbb{R}^4$.

Note: There exist maps $p : SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$

given by $p(A, B) = A \circ B$ (matrix multiplication)

(Recall: $\det(A \cdot B) = \det(A) \cdot \det(B)$.)

Additionally, $i : SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$ given

by $i(A) = A^{-1}$

(Recall: $\det(A^{-1}) = \dfrac{1}{\det(A)}$).

It is the case that both p and i are smooth maps.

Def| If X is a smooth manifold with a group operation $p: X \times X \to X$ s.t. the maps p and i ($i: X \to X$ s.t. $i(a) = a^{-1}$) are smooth, then X is a Lie group.

* Lie groups are super important!!!

~~Proposition~~

For many applications (such as algebraic geometry) we are interested in studying the set of zeros of some collection of real-value functions on a smooth manifold X.

The set up| $g_i : X \to \mathbb{R}$ are smooth functions

$$Z = \{ x \in X \mid g_1(x) = g_2(x) = \cdots = g_i(x) = 0 \}.$$

Note that $(dg_i)_x : T_x(X) \to \mathbb{R}$.

We say $g_1, \ldots, g_i$ are linearly independant at $x \in X$ if the set of vectors $(dg_1)_x, \ldots, (dg_i)_x$ are linearly independant.

Prop If $g_1, \dots, g_i$ are smooth real-valued functions on X and they are linearly independent on every point in Z, then Z is a submanifold of dimension $\dim(X) - i$.

Pf Define $g : X \to \mathbb{R}^i$ by $g(x) = (g_1(x), \dots, g_i(x))$

$$g^{-1}(0) = Z$$

Let $x \in g^{-1}(0)$

$$dg_x = \begin{bmatrix} - (dg_1)_x - \\ - (dg_2)_x - \\ \vdots \\ - (dg_i)_x - \end{bmatrix}$$

dg_x is onto iff $(dg_1)_x, \dots, (dg_i)_x$ are L.I.

Hence dg_x is onto and g is a submersion at x, so 0 is a regular value. By the preimage thm, $g^{-1}(0)$ is a submanifold of dimension $\dim X - i$. $\square$