Whitney Embedding Th^m 2 Lec. 15

<u>Th^m</u> Every k-dimensional smooth manifold admits a one-to-one immersion into $\mathbb{R}^{2k+1}$.

<u>Pf</u> Let X ~~$\subset \mathbb{R}^N$~~ be a smooth k-manifold and let $f: X \to \mathbb{R}^N$ be an embedding for $N > 2k+1$.

Define: $h: X \times X \times \mathbb{R} \to \mathbb{R}^N$ by $h(x,y,t) = t(f(x) - f(y))$

Define: $g: T(X) \to \mathbb{R}^N$ by $g((x,v)) = df_x(v)$.

Claim: h and g are smooth

<u>Pf</u> Exercise.

Claim: If $f: X \to Y$ is a smooth map and $\dim(X) < \dim(Y)$, then $f(X)$ has measure zero.

<u>Pf</u> Last time

By claim, both $Im(h)$ and $Im(g)$ have measure zero in $\mathbb{R}^N$. Hence, we can choose $a \in \mathbb{R}^N \setminus \{\vec{0}\}$ s.t. $a \notin Im(h)$ and $a \notin Im(g)$.

Define $\pi: \mathbb{R}^N \to H$ be the projection map where H is the $(N-1)$-dim'l subspace of $\mathbb{R}^N$ that is perpendicular to a.

Claim: $\pi \circ f : X \to H$ is one-to-one.

$\boxed{\text{Pf}}$ Suppose not. Let $x, y \in X$ s.t. $x \neq y$ and $\pi \circ f(x) = \pi \circ f(y)$.

Since f is an embedding, $f(x) \neq f(y)$.

Hence, $\exists\, t \in \mathbb{R} \setminus \{0\}$ s.t. $f(x) - f(y) = t \cdot a$.

$$\text{or} \quad 1/t\,(f(x) - f(y)) = a.$$

$$\text{or} \quad h(x, y, 1/t) = a.$$

However, this contradicts our choice of a. $\square$

Claim: $\pi \circ f : X \to H$ is an immersion.

$\boxed{\text{Pf}}$ Suppose not. There exists $x \in X$ and $v \in T_x(X)$ s.t. $v \neq \vec{0}$ and $\cancel{df_x(v)} = \vec{0}$. $d(\pi \circ f)_x(v) = \vec{0}$.

By chain rule, $\cancel{d\pi_{f}}\; d\pi_{f(x)} \circ df_x(v) = \vec{0}$.

Since π is linear, $\pi \circ df_x(v) = \vec{0}$.

So $df_x(v) = ta$ for some $t \in \mathbb{R}$.

* Note $t \neq 0$ since df_x is one-to-one.

$$1/t\; df_x(v) = a$$

$$df_x(1/t \cdot v) = a$$

$$g((x, 1/t \cdot v) = a \quad \divideontimes \quad \square$$

Thus $\pi \circ f : X \to H \cong \mathbb{R}^{N-1}$ is a one-to-one immersion.

By induction $\exists\, g : X \to \mathbb{R}^{2k+1}$, a one-to-one immersion. $\square$

Corrolary If X is a compact k-manifold, then X can be embedded in $\mathbb{R}^{2k+1}$.

Pf By theorem, there exists a one-to-one immersion $g: X \to \mathbb{R}^{2k+1}$.
Need to show g is proper. Let $C \subset \mathbb{R}^{2k+1}$ be compact. By H.B. C is closed & bounded. Thus $g^{-1}(C) \subset X$ is closed. Since closed subsets of compact sets are compact, then $g^{-1}(C)$ is compact. $\square$

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partitions of unity

Th^m Let X be ~~an arbitrary subset of $\mathbb{R}^N$~~ Manifold.
For any covering of X by open sets $\{U_\alpha\}$ in X, there exists a sequence of smooth functions $\{\theta_i\}_{i=1}^{\infty}$ s.t. $\theta_i : X \to \mathbb{R}$, called a partition of unity subordinate to $\{U_\alpha\}$, with the following properties

1) $0 \le \theta_i(x) \le 1$ for all $x \in X$ and all i.
2) $\forall x \in X$ $\exists U_x$ a nbh of x in X s.t. all but finitely many θ_i are non-zero on U_x
3) Each θ_i is zero outside of a closed subset of some U_α.
4) $\forall x \in X$ $\sum \theta_i(x) = 1$ (a finite sum!)

Pf In Book.

$\sum_{i=1}^{\infty}$

Corollary | Given any manifold X, there is a proper map $\rho : X \to \mathbb{R}$.

Pf | Let $\{U_\alpha\}$ be the collection of open sets in X that have compact closures. Let $\{\theta_i\}$ be a subordinate partition of unity.

Define $\rho : X \to \mathbb{R}$ by $\rho(x) = \sum_{i=1}^{\infty} i\,\theta_i(x)$.
Note ρ is smooth.

If $\rho(x) \leq j$, then at least one of $\theta_1(x), \ldots, \theta_j(x)$ is non-zero.

So; $\rho^{-1}([-j, j]) \subset \bigcup_{i=1}^{j} \{x \in X \mid \theta_i(x) \neq 0\}$

$$\subset \bigcup_{i=1}^{j} U_{\alpha_i}$$

But, $cl(\bigcup_{i=1}^{j} U_{\alpha_i})$ is compact by assumption and $\rho^{-1}([-j, j])$ is closed, by continuity.

So $\rho^{-1}([-j, j])$ is a closed subset of a compact set, so is compact.
By H.B. any compact subset of $\mathbb{R}$ is contained in $[-j, j]$ for some j. So, ρ is proper. $\square$