# Applications of Morse Theory to Knot Theory. Lec. 13

## Last time

**Thm (Morse Lemma)** Suppose $a \in \mathbb{R}^k$ is a non-degenerate critical point of a smooth map $f: \mathbb{R}^k \to \mathbb{R}$ and Let $H = (h_{ij})$ be the Hessian of $f$ at $a$. Suppose $H$ has $p$ positive eigenvalues and $n$ negative eigenvalues $(p+n=k)$, then there exist local coordinate systems s.t. $y_1, \dots, y_k$ s.t. $f(y_1, \dots, y_k) = f(a) + \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{k} y_i^2$.

~~Thm~~

**Def** ~~A function~~ A smooth map $f: X \to \mathbb{R}$ is morse if all critical points are ~~none~~ non degenerate.

**Thm** The set of smooth morse maps $f: X \to \mathbb{R}$ is open and dence in the space of smooth maps $f: X \to \mathbb{R}$ (i.e. non-morse maps are a set of measure zero).

## Knots

A **knot** is an embedding $f: S^1 \to \mathbb{R}^3$.

Two knots $f_1: S^1 \to \mathbb{R}^3$ and $f_2: S^1 \to \mathbb{R}^3$ are equivalent if There exists a smooth homotopy $F: \mathbb{R}^3 \times I \to \mathbb{R}^3$ s.t.

$F(x,t)$ is a diffeomorphism for all fixed $t$

$$F(x,0) = id_{\mathbb{R}^3}$$

and $$F(f_1(s), 1) = f_2(s).$$

We say $f_1$ is ambient isotopic to $f_2$.

Let $p: \mathbb{R}^3 \to \mathbb{R}$ be $p(x,y,z) = z$ be the standard projection map.

**Cor** Every knot $f: S^1 \to \mathbb{R}^3$ is ambient isotopic to a knot $f_2: S^1 \to \mathbb{R}^3$ s.t. $p \circ f_2: S^1 \to \mathbb{R}$ is morse.

**Pf** Appeal to the fact that non-morse are a set of measure zero.

Lemma | If $f: S^1 \to \mathbb{R}$ is a morse function then the set of critical points in $S^1$ is a finite set.

Pf | ~~From last time, every critical point $x \in S^1$ has a nbh $U_x$ s.t. $x$ is the only~~ critical point ~~in $U_x$.~~ Suppose the set of non-degenerate critical points is an infinite set. Since $S^1$ is a compact subset of $\mathbb{R}^N$, then we can construct a convergent sub sequence of these points s.t. $\lim_{n\to\infty} x_{i_n} = y \in S^1$.

If $y$ is a ~~non-degenerate critical point~~ critical point, then $y$ is a non-degenerate crit point. By the lemma from last time, $\exists\, U_y$ s.t. $y$ is the unique crit point in $U_y$ $\divideontimes$.

If $y$ is a regular point, then ~~$df_y$~~ $df_y$ has rank 1

$$df_y = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_k} \right]$$

Since $\frac{\partial f}{\partial x_i}$ is continuous $\exists$ an open ~~inter~~ set $U_y$ in $X$ s.t. ...

$\divideontimes$

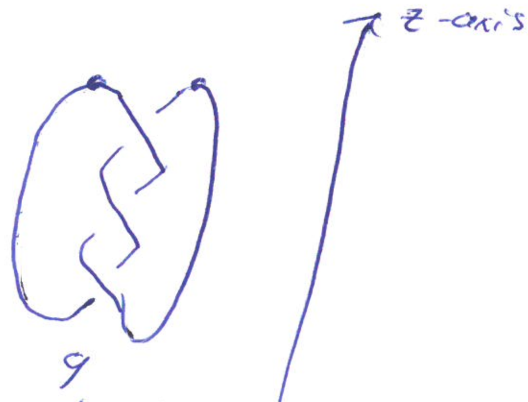
By morse Theory, every critical point of $f: S^1 \to \mathbb{R}$ is locally modled on $f(x) = f(a) + x^2$ or

$$f(x) = f(a) - x^2$$

ie. a local min or a local max.

<u>Def</u> The bridge number of a knot type $K = [f]$ is the minimal number of maxima of any morse embedding $g \in [f]$. (Denoted $\beta(K)$)

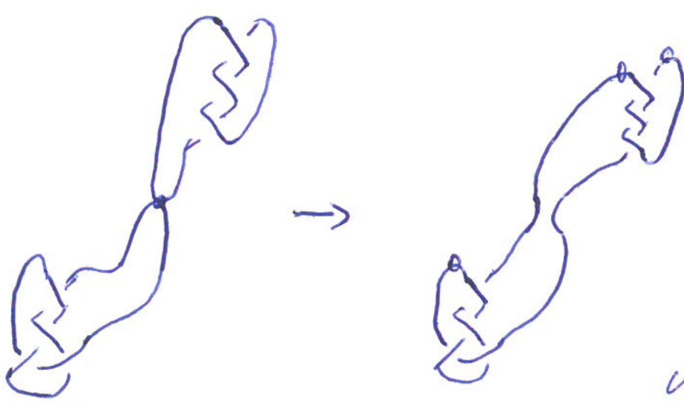
<u>Ex</u>




$\beta(g) = \beta([g]) = 2$

<u>Def</u> Connected sum of knots

Given morse embeddings $K_1$ and $K_2$



well define (up to worrying about

$K$ $L$ $K \# L$

**Thm** (Schubert ('54))

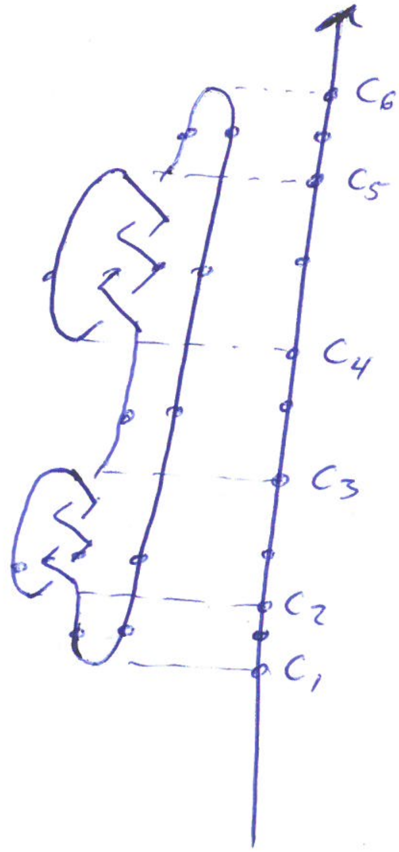
$$\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1$$

$\leq$ easy

$\geq$ hard!

**Width**

Let $f: S^1 \to \mathbb{R}^3$ be ~~an~~ a knot s.t. $p \circ f: S^1 \to \mathbb{R}$ is morse with isolated critical values. $c_1 \leq \cdots \leq c_n$




~~Let~~ Pick $c_i \leq r_i \leq c_{i+1}$ be representative regular values

$$w(g) = \sum_{i=1}^{n-1} |(p \circ f)^{-1}(r_i)|$$

Define $w([g]) = \min_{\substack{g \in [g] \\ g \text{ is morse}}} w(g)$.

$w([g])$ is known as the width of a knot.

Question: $w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$ ?

$\leq$ easy

$\geq \leftarrow$ known for large classes of knots (Rieck & Sedgwick)

$\geq \max(w(K_1), w(K_2))$ Scharlemann & ~~Abo.~~ Schultens.