Lec. 12

Annoucements

- H.W. Due Thursday (Friday Morning).
- Want to move exam? Current March 10th, Thurs. Tues 15th or 17th Thurs

Recall from last time

Def Given a smooth map $f: X \to Y$, $y \in Y$ is a <u>critical value</u> of f if y is <u>not</u> a regular value. Moreover, $x \in X$ is a <u>critical point</u> if df_x is not onto and is a <u>regular point</u> if df_x is onto.

Thm (Sard's Theorem) If $f: X \to Y$ is a smooth map, then the set of critical values ~~of~~ of f have measure 0 in Y.

(Important! It is <u>not</u> true that the set of critical points have measure 0 in X)

Ex $f: \mathbb{R}^2 \to \mathbb{R}$ via $f(x, y, \cancel{z}) = \vec{0}$.

Every non-zero value of $\mathbb{R}$ is vacuously a regular value of f. 0 is a critical value. All points in $\mathbb{R}^2$ are critical points.

nondegenerate critical points

We want to study the local behavior of smooth maps $f: X \longrightarrow \mathbb{R}$.

First Consider $f: \mathbb{R}^k \to \mathbb{R}$.

Suppose $x \in \mathbb{R}^k$ is a critical point (i.e. $df_x = 0$)

Define the Hessian matrix ~~at x to be~~

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \ddots & \vdots \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_k} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_k} \end{bmatrix}$$

If H is nonsingular at x, we say x is a <u>non degenerate</u> critical point.

<u>Lemma</u> If $x \in \mathbb{R}^k$ is a non-degenerate critical point of a smooth map $f: \mathbb{R}^k \longrightarrow \mathbb{R}$, then there exists an open nbh of x in $\mathbb{R}^k$ s.t. $f|_U$ has a unique critical point at x.

Pf| Define $g: \mathbb{R}^k \to \mathbb{R}^k$ via

$$g(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right)$$

Since $df_x = \cancel{g(x)} \left[\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right]$,

then $df_x = 0$ iff $g(x) = (0, \ldots, 0)$.

$$dg_x = \left[\quad \right] = H$$

Since x is a non degenerate critical point, then dg_x is a vector space isomorphism.

By the inverse function theorem, since dg_x is a vector space isomorphism, then g is a local diffeomorphism at x.

Hence, there exists an open nbh U of x in $\mathbb{R}^k$ s.t.

$$g|_U(y) = 0 \Rightarrow y = x.$$

Equivalently, $f|_U$ has a unique critical point at x. $\square$

Thm (Morse Lemma)| Suppose $a \in \mathbb{R}^k$ is a non degenerate critical point of $f: \mathbb{R}^k \to \mathbb{R}$. and let $H = (h_{ij})$ be the Hessian of f at a.

Then there exists a local coordinate system $\{x_1, \ldots, x_k\}$ around a s.t. $f = f(a) + \sum h_{ij} x_i x_j$ near a.

Moreover, since H is a real, symetric, invertable $k \times k$ matrix, H will have p positive eigen values and n negative eigen values s.t. $p+n=k$.

From linear algebra, real symetric matrices are diagonalizable. there exists yet another coordinate system $y_1, \ldots, y_k$ s.t.

$$f = f(a) + \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{n} y_i^2$$

Lets investigate the morse lemma for functions

$f : \mathbb{R}^2 \to \mathbb{R}$.

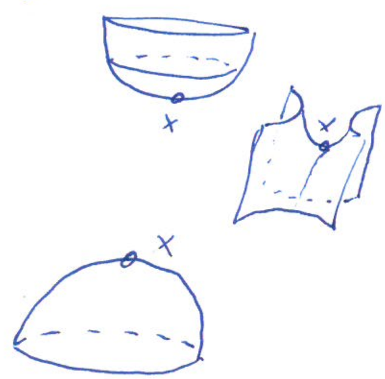
Suppose $f((0,0)) = 0$

Then $f((x,y)) = f((0,0)) + \sum_{i=1}^{2} x^2 + y^2$

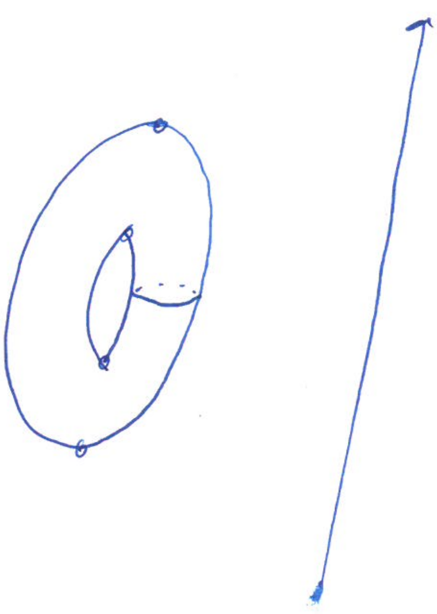
$= 0 + x^2 + y^2$ min

$+x^2 - y^2$ saddle

or $-x^2 - y^2$ max



Ex



non degenerate critical points for smooth maps

$f: X \to \mathbb{R}$.

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{R} \\ U \arrow[u, "\phi"', "\phi(0)=x"] & \end{tikzcd}$$

x is a critical point of f implies 0 is a crit. pt. of $f \circ \phi$.

Say x is a non degenerate crit point of $f: X \to \mathbb{R}$ if x is a nondegenerate crit point of $f \circ \phi$.

Problem: Suppose $\gamma: V \to X$ is another parameterization about $x \in X$ s.t. $\gamma(0) = x$. After restricting the domain & range of ϕ and γ we can assume $\phi^{-1} \circ \gamma$ and $\gamma^{-1} \circ \phi$ are diffeomorphisms.

Since $f \circ \gamma = f \circ \phi \circ (\phi^{-1} \circ \gamma)$, then we must show...

<u>Lemma</u>| Suppose $f: \mathbb{R}^k \to \mathbb{R}$ has a non-degenerate critical point at 0 and let $g: \mathbb{R}^k \to \mathbb{R}^k$ be a diffeomorphism with $g(0) = 0$. Then $f \circ g$ also has a non degenerate crit value at 0.

<u>Pf</u>| Terrible