MATH 550A - Spring 2014
Midterm Two

Name: Solutions

PLEASE WRITE LEGIBLY AND EXPLAIN YOUR STEPS CAREFULLY, USING COMPLETE SENTENCES. No books and no notes. Remember to put your name at the top of this page. Good luck!

Problem	Score (out of)
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
Total	(50)

1.(10pts)

A) State the Heine-Borel theorem.

A subspace of $\mathbb{R}^n$ is compact iff it is closed and bounded.

B) Prove the Heine-Borel theorem. Carefully state any theorems that you use in the proof.

$\Rightarrow$ Suppose $X \subset \mathbb{R}^n$ is compact.

Let $\overset{\circ}{B}_\varepsilon(\vec{x}) = \{\vec{y} \in \mathbb{R}^n \mid d(\vec{x}, \vec{y}) < \varepsilon\}$

Note that $\{B_n(\vec{0})\}_{n \in \mathbb{Z}^+}$ is an open cover of X.

Since X is compact, there is a finit subcover $\{B_{n_i}(\vec{0})\}_{i=1}^{m}$ where $n_1 < n_2 < \cdots < n_m$. Hence, $X \subset B_{n_m}(\vec{0})$ and X is bounded.

Prop 1: A compact subset of a Hausdorff space is closed.

By Prop 1, X is closed. $\square$

$\Leftarrow$ Suppose $X \subset \mathbb{R}^n$ is closed and bounded.

Since X is bounded, $\exists M \in \mathbb{Z}^+$ s.t. $X \subset \prod_{i=1}^{n} [-M, M]$.

Since closed intervals in $\mathbb{R}$ are compact and since the product of compact spaces are compact, then $\prod_{i=1}^{n} [-M, M]$ is compact. Since X is closed in $\mathbb{R}^n$, then X is closed in $\prod_{i=1}^{n} [-M, M]$. Since closed subsets of compact spaces are compact, then X is compact. $\square$

2.(10pts)

A) Define the finite complement topology on a set X.

$$\tau = \{ U \subset X \mid X - U \text{ is a finite set} \}$$

B) Prove that $\mathbb{R}$ with the finite complement topology is compact.

Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of $(\mathbb{R}, \tau)$.

Let U_0 be an element of this cover. Since $U_0 \in \tau$, then ~~$\Leftrightarrow$~~ $\mathbb{R} - U_0 = \{x_1, \dots, x_n\}$. Since $\{U_\alpha\}_{\alpha \in J}$ is a cover then there exists $U_i \in \{U_\alpha\}_{\alpha \in J}$ s.t. $x_i \in U_i$ for each $1 \le i \le n$.

Claim: $\{U_i\}_{i=0}^{n}$ is a finite subcover

This set is obviously finite.

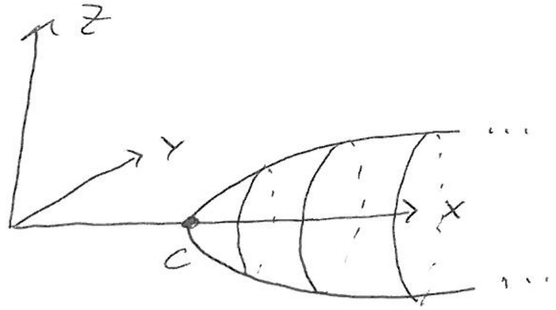
Let $x \in \mathbb{R}$. If $x \in U_0$, then we are done. If $x \notin U_0$ then $x = x_i$ for $1 \le i \le n$ and $x \in U_i$. Hence $X \subset \bigcup_{i=0}^{n} U_i$ and $\{U_i\}_{i=0}^{n}$ is a finite subcover. $\square$

3.(10pts) Let X be the topological space $\mathbb{R}^3$ with the product topology. Define an equivalence relation on X by setting

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \quad \text{if} \quad x_1 - ((y_1)^2 + (z_1)^2) = x_2 - ((y_2)^2 + (z_2)^2).$$

Let X^* be the quotient space $X/\sim$. To what familiar space is X^* homeomorphic? For full credit you must give a convincing argument.

The equivalence classes of $\sim$ are graphs of the function $x = y^2 + z^2 + c$ where c can be any real number. These graphs are paraboloids opening along the x-axis



The x-axis meets each of these equivalence classes in exactly one point $(c,0,0)$.

Hence $X/\sim = \mathbb{R}$ as a set. and $p: X \to \mathbb{R}$ via $p((x,y,z)) = x - y^2 - z^2$.

let (a,b) be an open interval in $\mathbb{R}$.

$p^{-1}((a,b))$ is the collection of points in $\mathbb{R}^3$ strictly between the graphs of disjiont continuous functions ($x - y^2 - z^2 = a$ and $x - y^2 - z^2 = b$), which is an open set in $\mathbb{R}^3$. Hence open intervals in $\mathbb{R}$ are open.

Thus, it is a good bet that $\mathbb{R}$ has the standard topology.

4. (10pts)

A) Define a separation of a topological space X.

Sets $A, B \subset X$ form a separation $X = A|B$ if all of the following hold.

① $A \cap B = \emptyset$

② $A \cup B = X$

③ A and B are open

④ $A \neq \emptyset$ and $B \neq \emptyset$.

B)Let X and Y be topological spaces such that X is connected. Let $f : X \to Y$ be a continuous, surjective map. Show that Y is connected.

Let X be a connected top. space and let $f: X \to Y$ be a continuous surjective map. Suppose, to form a contradiction, that Y has a separation, $Y = A|B$.

Claim | $X = f^{-1}(A) | f^{-1}(B)$.

③ Since A and B are open in Y, and f is continuous $f^{-1}(A)$ and $f^{-1}(B)$ are open in X.

④ Since A and B are non empty and f is surjective, then both $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty

② Note

$$A \cup B = Y$$

$$f^{-1}(A \cup B) = f^{-1}(Y)$$

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(Y) = X$$

① If $x \in f^{-1}(A) \cap f^{-1}(B)$ then $f(x) \in A$ and $f(x) \in B$. But $A \cap B = \emptyset$ hence x can not exist. Thus $f^{-1}(A) \cap f^{-1}(B) = \emptyset$.

~~Since X has~~

However, X having a separation contradicts X being connected. So, Y is connected. $\square$

5. (10pts) Let $f : \mathbb{R} \to \prod_{i\in\mathbb{Z}+} \mathbb{R}$ be the map given by $f(t) = (t, t, t, ...)$.

A) Prove or disprove that f is continuous when $\prod_{i\in\mathbb{Z}+} \mathbb{R}$ has the product topology and $\mathbb{R}$ has the standard topology.

By a lemma from class, it suffices to show that f^{-1} of a subbasis element of $(\mathbb{R}^\omega, \text{prod})$ is open in $\mathbb{R}$. Recall, an arbitrary sub basis element is of the form $\pi_i^{-1}(U)$ where U is open in $\mathbb{R}$ and $\pi_i : \mathbb{R}^\omega \to \mathbb{R}$ via $\pi_i((x_1, x_2, x_3, \dots)) = x_i$. ~~$t \in f^{-1}(\pi_i^{-1}(U))$~~

$f^{-1}(\pi_i^{-1}(U)) = (\pi_i \circ f)^{-1}(U)$. But $\pi_i \circ f(t) = t$, so, $(\pi_i \circ f)^{-1}(U) = U$ which is open by hypothesis. Thus f is continuous. $\square$

B) Prove or disprove that f is continuous when $\prod_{i\in\mathbb{Z}+} \mathbb{R}$ has the box topology and $\mathbb{R}$ has the standard topology.

$A = \prod_{i=1}^{\infty} (\frac{-1}{i}, \frac{1}{i})$ is open in $(\mathbb{R}^\omega, \text{box})$ since it is the product of open sets. Examine $f^{-1}(A)$. $t \in f^{-1}(A)$ iff $t \in (-1, 1)$ and $t \in (\frac{-1}{2}, \frac{1}{2})$ and $t \in (\frac{-1}{3}, \frac{1}{3}) \cdots$ Hence, $t \in f^{-1}(A)$ iff $t \in \bigcap_{i=1}^{\infty} (\frac{-1}{i}, \frac{1}{i}) = \{0\}$. However, $\{0\}$ is ~~clos~~ not open in $\mathbb{R}$, so f is not continuous. $\square$