Topology I Day 2

LAS-151

Outline

- Basis for a topology.

- Announcements
- OH today cancelled
- Homework posted online

Analogy:

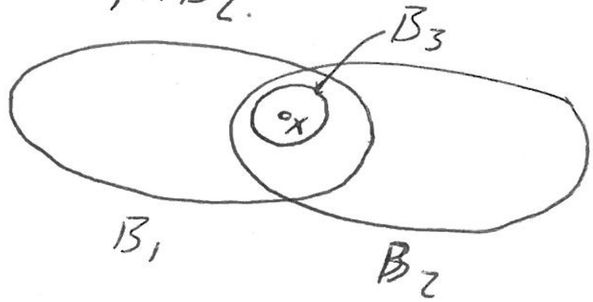
In linear algebra any vector is a linear combination of basis vectors

In topology any open set is a union of basic open sets

<u>Defn</u> Let X be a set. A <u>basis for a topology</u> on X is a collection of subsets of X $\mathcal{B}$ s.t.

① $\forall x \in X$ $\exists B \in \mathcal{B}$ s.t. $x \in B$.

② If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.



How to use a basis $\mathcal{B}$ to define a topology $\mathcal{T}$. Declare $U \subset X$ to be open if $\forall x \in U$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.
$\mathcal{T}$ is the topology generated by $\mathcal{B}$.

Notation: Each element $B \subset \mathcal{B}$ is a <u>basis element</u>

Observe: Every basis element is open.

Ex| $\mathbb{R}$ ~~with the standard topology~~ has basis $\mathcal{B} = \{$all open intervals$\}$

① $\forall x \in \mathbb{R}\ \exists (x-1, x+1)$ s.t. $x \in (x-1, x+1)$.

② If (a,b) and (c,d) are elements of $\mathcal{B}$ and $x \in (a,b) \cap (c,d)$, the $x \in (c,b) \subset (a,b) \cap (c,d)$.

Hence $\mathcal{B}$ is a basis.

- The topology on $\mathbb{R}$ generated by $\mathcal{B}$ is called the standard topology.

Ex| $\mathbb{R}^2$ has basis $\mathcal{B} = \{$ interiors of all circles $\}$.

Ex| $\mathbb{R}^2$ has basis $\mathcal{B} = \{$interiors of all rectangles with sides parallel to the axes$\}$

* Later we with show that these basis generate the same topology.

Lemma 1: The topology τ generated by a basis $\mathcal{B}$ is in fact a topology for X.

Proof: Claim 1: $\emptyset$ is open

Since $\emptyset$ contains no elements it is open vacuously.

Claim 2: X is open

Let $x \in X$. By def of basis $\exists B \in \mathcal{B}$ s.t. $x \in B \subset X$.

So, X is open.

Claim 3: Let $\{\mathcal{U}_\alpha\}$ be any collection of open sets, then $\bigcup_{\alpha \in A} \mathcal{U}_\alpha$ is open.

Let $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$. Let $x \in \bigcup_\alpha \mathcal{U}_\alpha$. $x \in \mathcal{U}_\beta$ for some fixed β. Since $\mathcal{U}_\beta$ is open, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset \mathcal{U}_\beta$. Hence, $x \in B \subset \bigcup_\alpha \mathcal{U}_\alpha$ and $\bigcup_\alpha \mathcal{U}_\alpha$ is open.

Claim 4: Let $\{\mathcal{U}_i\}_{i=1}^n$ be a collection of open sets, then $\bigcap_{i=1}^n \mathcal{U}_i$ is open.

If $\bigcap_{i=1}^n \mathcal{U}_i = \emptyset$, then we are done. Let $x \in \bigcap_{i=1}^n \mathcal{U}_i$. By definition of open, $\exists B_i \in \mathcal{B}$ s.t. $x \in B_i \subset \mathcal{U}_i$ for $1 \le i \le n$.

If $n=2$, by def of basis $\exists B_{1,2} \in \mathcal{B}$ s.t. $x \in B_{1,2} \subset B_1 \cap B_2$. Hence $\bigcap_{i=1}^n \mathcal{U}_i$ is open.

Suppose $\bigcap_{i=1}^n \mathcal{U}_i$ is open for $n=k$.

Suppose $n=k+1$. Since $\bigcap_{i=1}^k \mathcal{U}_i$ is open, $\exists B_* \in \mathcal{B}$ s.t. $x \in B_* \subset \bigcap_{i=1}^k \mathcal{U}_i$. By defn of Basis $\exists B \in \mathcal{B}$ s.t. $x \in B \subset B_* \cap B_{k+1} \subset \bigcap_{i=1}^n \mathcal{U}_i$

Hence, by induction, $\bigcap_{i=1}^{n} \mathcal{U}_i$ is open.

By claims 1, 2, 3, 4 τ is infact a topology. $\square$

Lemma 2: Let $\mathcal{B}$ be a basis for a topology τ on X. Then $\mathcal{U} \subset X$ is open iff $\mathcal{U}$ is the union of elements of $\mathcal{B}$.

<u>Pf</u>| Suppose $\mathcal{U}$ is open. By def. of open, $\forall x \in \mathcal{U}$ $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset \mathcal{U}$.

Claim: $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x$

Let $x \in \mathcal{U}$, then $x \in B_x \subset \bigcup_{x \in \mathcal{U}} B_x$. So, $\mathcal{U} \subset \bigcup_{x \in \mathcal{U}} B_x$

Let $x \in \bigcup_{x \in \mathcal{U}} B_x$, then $x \in B_x \subset \mathcal{U}$. So, $\bigcup_{x \in \mathcal{U}} B_x \subset \mathcal{U}$.

Next, suppose $\mathcal{U} = \bigcup_{A \in \mathcal{A}} B_A$ where $B_A \in \mathcal{B}$.

Let $x \in \mathcal{U} = \bigcup_{A \in \mathcal{A}} B_A$. Hence, $x \in B_A^{\subset \mathcal{U}}$ for some $A \in \mathcal{A}$.

Hence, $\mathcal{U}$ is open by definition. $\square$

Lemma 3: Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C}$ be a collection of open sets s.t.

(*) Given $U \subset X$ open and $x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$.

Then $\mathcal{C}$ is a basis <u>for the topology $\mathcal{T}$</u>.

<u>Proof</u>| Step 1: Show $\mathcal{C}$ is a basis for some topology $\mathcal{T}'$.

① Let $x \in X$, since X is open, then, by (*), $\exists C \in \mathcal{C}$ s.t. $x \in C \in \mathcal{C}$.

② Let $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{C}$.

Since B_1 and B_2 are open by hypothesis, then $B_1 \cap B_2$ is open. By (*) $\exists C \in \mathcal{C}$ s.t. $x \in C \subset B_1 \cap B_2$.

Hence, by ① and ② $\mathcal{C}$ is a basis.

Step 2: Show $\mathcal{T} = \mathcal{T}'$

<u>$\subseteq$</u>| Let $U \in \mathcal{T}$. Let $x \in U$. By (*), $\exists C \in \mathcal{C}$ s.t. $x \in C \subset U$. By def. of open w.r.t. a basis U is open in $\mathcal{T}'$, so $U \in \mathcal{T}'$.

<u>$\supseteq$</u>| Let $U \in \mathcal{T}'$. By def. of open w.r.t a basis $\forall x \in U \; \exists C_x \in \mathcal{C}$ s.t. $x \in C_x \subset U$.

Hence, $U = \bigcup_{x \in U} C_x$ (the union of open sets in $\mathcal{T}$).

Thus, $U \in \mathcal{T}$.

So, $\mathcal{T} = \mathcal{T}'$ $\square$

Lemma 4 | Let X be a set and $\mathcal{B}, \mathcal{B}'$ be bases for topologies τ and τ'. T.F.A.E.

1) $\tau' \supset \tau$ (τ' is finer than τ)

2) Given any $x \in X$ and $B \in \mathcal{B}$ s.t. $x \in B$, $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

Pf | In Munkres.

Applications |

Let τ' be a topology on $\mathbb{R}$ induced by basis $\mathcal{B}' = \{[a,b) : a<b\}$

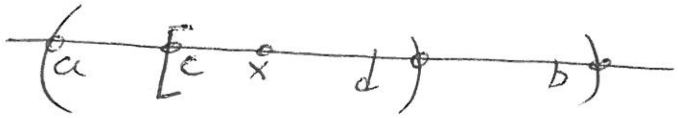
$\mathbb{R}_\ell = (\mathbb{R}, \tau')$ is the lower limit topology.

Claim: $\mathbb{R}_\ell$ is finer than $\mathbb{R}$.

Pf | Let $x \in \mathbb{R}$ and $(a,b) \in \mathcal{B}$ s.t. $x \in (a,b)$.

$$x \in \left[\cancel{a+}\frac{x+a}{2}, \cancel{b-}\frac{x+b}{2}\right) \not\subset,$$

$$[c,d) \subset (a,b)$$



Hence, by Lemma 4; $\mathbb{R}_\ell$ is finer than $\mathbb{R}$.

(In fact it is strictly finer).

Ex Let $K = \{ \frac{1}{n} : n \in \mathbb{Z}^+ \}$

$\mathcal{B}'' = \{ (a,b) \mid a<b \} \cup \{ (a,b) - K \mid a<b \}$

is a basis for $\mathbb{R}$ with topollogy τ''.

Denote $\mathbb{R}_K = (\mathbb{R}, \tau'')$

Q: How does $\mathbb{R}_\ell, \mathbb{R}, \mathbb{R}_K$ all compare?

Outline
- Examples of interesting topologies
- Order topology
- Subspace topology
- Product topology

Example | Let $X = \mathbb{R}$. $\mathcal{B}' = \{[a,b) \mid a < b\}$

$\mathcal{B}'$ is a basis for $\mathbb{R}$ generating a topology τ'.

This is the "lower-limit topology" $\mathbb{R}_\ell = (\mathbb{R}, \tau')$.

Recall: The standard topology on $\mathbb{R}$ is generated by the basis of open intervals. ($\mathbb{R}_s$)

Recall Lemma 4 from last time |

Let X be a set with basis $\mathcal{B}$ and $\mathcal{B}'$ that generate topologies τ and τ'. T.F.A.E.

1) $\tau' \geq \tau$ (τ' is finer than τ)

2) Given any $x \in X$ and $B \in \mathcal{B}$ containing x, $\exists\, B' \in \mathcal{B}'$ with $x \in B' \subset B$.

Example | Show $\mathbb{R}_\ell$ is finer than $\mathbb{R}_s$.

~~Let $x \in (a,b)$.~~ Let $x \in \mathbb{R}_s$ s.t. $x \in (a,b) \in \mathcal{B}$

$$x \in \left[a + \tfrac{x-a}{2},\, b\right) \subset (a,b).$$

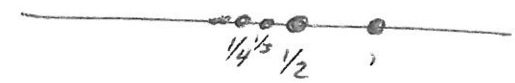
Hence, by lemma 4, τ' is finer than τ.

Is $\mathbb{R}_\ell$ strictly finer than $\mathbb{R}_s$?

this is equivalent to

Are there elements of τ' that are not contained in τ?

Yes, $[0,1)$.

<u>Example</u>| Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$ 

Let $\mathcal{B}'' = \{(a,b) \mid a<b\} \cup \{(a,b)-K \mid a<b\}$

$\mathcal{B}''$ is a basis for a topology τ''

$\mathbb{R}_K = (\mathbb{R}, \tau'')$

<u>Exercise</u>| Use Lemma 4 from last time to show τ'' is finer than τ.

<u>Def</u>| A sub-basis for a topology on a set X is a collection S s.t. $\bigcup_{A \in S} A = X$ and S generates a topology τ by taking all possible unions of finite intersections of elements of S.

<u>Order Topology</u>|

A set X is an <u>ordered set</u> if it has a relation $<$ s.t.

1) If $a, b \in X$ s.t. $a \neq b$, then $a<b$ or $b<a$.
2) "$a<a$" holds for <u>no</u> $a \in X$
3) If $a<b$ and $b<c$, then $a<c$.

Note: If X is an ordered set, then define (a,b), $[a,b)$, $(a,b]$ and $[a,b]$ in the obvious way.

Def If X is an ~~open s~~ ordered set, the order topology on X is generated by a basis $\mathcal{B}$ consisting of

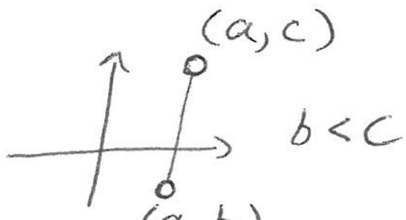
- all open intervals (a, b)
- intervals $[a_0, b)$ where a_0 is a least element (if it exists)
- intervals $(a, b_0]$ where b_0 is a greatest element (if it exists).

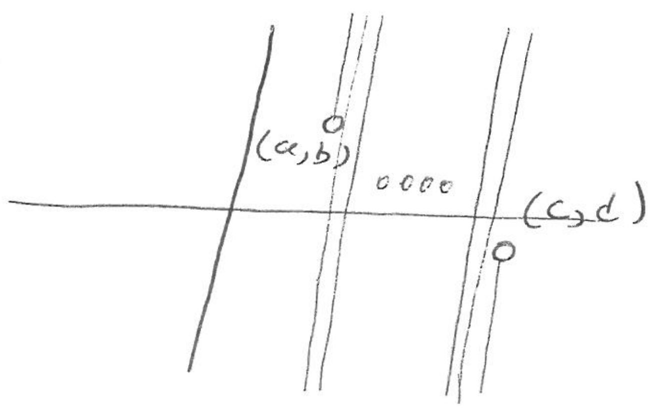
Question: What is the order topology on $\mathbb{R}$ with the standard $<$ opperation.

Example $\mathbb{R} \times \mathbb{R}$ has a dictionary order where

$$(a,b) < (c,d) \text{ if } a < c$$

$$\text{or } a = c \text{ and } b < d.$$





Subspace Topology

Let (X, τ) be a topological space with $Y \subset X$. Define a topology τ_Y on Y by declaring $V \subset Y$ is open if $V = Y \cap U$ for $U \in \tau$.

i.e. $\tau_Y = \{U \cap Y \mid U \in \tau\}$

Verify that τ_Y is a topology on Y.

① $\emptyset \cap Y = \emptyset \in \tau_Y$

② $X \cap Y = Y \in \tau_Y$

③ Let $\{U_\alpha \cap Y\}_{\alpha \in A}$ be a collection of sets in τ_Y

$$\bigcup_{\alpha \in A} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in A} U_\alpha\right) \cap Y.$$

Since $\bigcup_{\alpha \in A} U_\alpha$ is open in X then $\bigcup_{\alpha \in A} (U_\alpha \cap Y) \in \tau_Y$.

④ Let $\{U_i \cap Y\}_{i=1}^{n}$ be a collection of sets in τ_Y.

$$\bigcap_{i=1}^{n} (U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y$$

Since $\bigcap_{i=1}^{n} U_i$ is open in X, then $\bigcap_{i=1}^{n} (U_i \cap Y)$ is open in τ_Y.

Example | Let $Y = [0,1]$ and $X = \mathbb{R}$.
Give $\mathbb{R}$ the standard topology.
Give Y the subspace topology inherited from $\mathbb{R}$.
Is there an open set in Y that is not open in $\mathbb{R}$?

Question: What topology does $\mathbb{Z} \subset \mathbb{R}_s$ inherit?

Product Topology

Def | Given topological spaces (X, τ_X) and (Y, τ_Y) the product topology on $X \times Y$ is give by the basis

$$\mathcal{B} = \{U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y\}$$

Verify that $\mathcal{B}$ is a basis.

① Let $x \in X \times Y$

Since $X \times Y$ is a basis element,

$x \in X \times Y \in \mathcal{B}$.

② Let $U_1 \times V_1$ and $U_2 \times V_2$ be basis elements s.t. $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$

Fact: $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.

~~Hence~~ Since $U_1 \cap U_2$ is open in X and

$V_1 \cap V_2$ is open in Y, then

$(U_1 \cap U_2) \times (V_1 \cap V_2)$ is a basis element.

Thus, $x \in (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$. $\square$

Fact if $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, then $\mathcal{B} = \{ B_1 \times B_2 \mid B_1 \in \mathcal{B}_X \text{ and } B_2 \in \mathcal{B}_Y \}$ is also a basis for the product topology.

Def $\pi_1 : X \times Y \to X \quad \pi_1((x, y)) = x$

$\pi_2 : X \times Y \to Y \quad \pi_2((x, y)) = y$

These are the projection maps.

Topology Day 4

Outline

- Quick Review
- Product topology
- Closed Sets
- Closure
- Limit points

- Review

  Q: How do we define the topology on $\mathbb{R}_\ell$?

  Q: Given an ordered set X, how do we define the ordered topology on X?

  Q: Given a top. space X and $Y \subset X$, how do we define the subspace topology on Y.

Product topology

Def Given top. spaces (X, τ_X) and (Y, τ_Y), the product topology on $X \times Y$ is given by the basis $\mathcal{B} = \{U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y\}$.

Claim: $\mathcal{B}$ is a basis for $X \times Y$.

① Show for every $(x, y) \in X \times Y$, $\exists B \in \mathcal{B}$ s.t. $(x, y) \in B$.

Pf Let $(x, y) \in X \times Y$. In particular $x \in X$ and $y \in Y$. Hence $(x, y) \in X \times Y \in \mathcal{B}$. $\square$

(2) Given $B_1, B_2 \in \mathcal{B}$ s.t. $(x,y) \in B_1 \cap B_2$, then $\exists B_* \in \mathcal{B}$ s.t. $(x,y) \in B_* \subset B_1 \cap B_2$.

Pf| $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ s.t. $U_1, U_2 \in \mathcal{T}_X$ and $V_1, V_2 \in \mathcal{T}_Y$.

However, $U_1 \cap U_2 \in \mathcal{T}_X$ and $x \in U_1 \cap U_2$.

Similarly, $V_1 \cap V_2 \in \mathcal{T}_Y$ and $y \in V_1 \cap V_2$.

Fact: $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

Hence, $(x,y) \in (U_1 \cap U_2) \times (V_1 \cap V_2) = (U_1 \times V_1) \cap (U_2 \times V_2)$.

Since, $U_1 \cap U_2 \in \mathcal{T}_X$ and $V_1 \cap \cancel{U_2} V_2 \in \mathcal{T}_Y$, then

$(x,y) \in (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$. $\square$

Exersice| If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y then $\mathcal{B} = \{U \times V \mid U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$.

Def| $\pi_1 : X \times Y \to X$ s.t. $\pi_1((x,y)) = x$

$\pi_2 : X \times Y \to Y$ s.t. $\pi_2((x,y)) = y$

There are projection maps.

Recall| A sub-basis for a topology on a set X is a collection of subsets S s.t. $\bigcup_{A \in S} A = X$ and S generates a topology by taking all possible unions of all finite intersections of elements of S.

Lemma: $S = \{\pi_1^{-1}(U) \mid U \in \tau_X\} \cup \{\pi_2^{-1}(V) \mid V \in \tau_Y\}$ is a sub basis of the product topology on $X \times Y$.

Pf| Notice that $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$.
Hence the basis for the product topology is contained in the set of all finite intersections of S.
Since all open sets are a union of basis elements, all open sets of $X \times Y$ are generated by taking arbitrary unions of finite intersections of S.

Q: How do the subspace & product topologies interact?

~~Prop~~ Given top. spaces X and Y s.t. $A \subset X$ and $B \subset Y$

Equip A and B with the subspace topology and Equip $A \times B$ with the product topology. Call this topology on $A \times B$ τ_X.

Equip $X \times Y$ with the product topology and Equip $A \times B$ with the subspace top. Call this top. on $A \times B$ τ_S.

Claim: $\tau_X = \tau_S$.

Pf| We will show that the bases

$\mathcal{B}_X = \{(U \cap A) \times (V \cap B) \mid U \in \tau_X, V \in \tau_Y\}$ and

$\mathcal{B}_S = \{(U \times V) \cap (A \times B) \mid U \in \tau_X, V \in \tau_Y\}$ are equal.

Set theory fact $(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B)$.

Hence, $\mathcal{B}_X = \mathcal{B}_S$.

Since, $\mathcal{B}_X = \mathcal{B}_S$, then $\tau_X = \tau_S$. $\square$

Closed Sets

<u>Def</u> If X is a top. space, $A \subset X$ is closed if $X - A$ is open.

* These are dual to, but just as important as open sets.

Examples

① $\emptyset$ and X are closed

② In $\mathbb{R}_S$, $[a,b]$ is closed
(since $\mathbb{R} - [a,b] = (-\infty, a) \cup (b, +\infty)$)

③ In the discrete topology, ever subset is closed
(since every subset is open)

④ In the finite complement topology,
all finite sets are closed.

<u>Def</u> Let τ_f be the finite-complement topology
on X if $\tau_f = \{X - A \mid \text{where } A \text{ is a finite subset of } X\}$.

⑤ In the product topology, if $A \subset X$ is closed and
$B \subset Y$ is closed then $A \times B \subset X \times Y$ is closed
(why: $X - A$ is open in X and $Y - B$ is open in Y
So, $(X - A) \times Y$ and $X \times (Y - B)$ are
open in $X \times Y$. Hence, $(X-A) \times Y \cup X \times (Y - B) = (X \times Y) - (A \times B)$
is open in $X \times Y$.)

<u>Example</u> Equip $Y = [0,1] \cup (2,3)$ with the subspace topology for $Y \subset \mathbb{R}_s$.

Is $[0,1]$ open in Y?

Is $(2,3)$ open in Y?

Is $[0,1]$ ~~open~~ closed in Y?

Is $(2,3)$ closed in Y?

Some key facts about closed sets.

<u>Prop</u> Let X be a top. space.

0) X and $\emptyset$ are closed sets

1) Arbitrary intersections of closed sets are closed

2) finite unions of closed sets are closed.

<u>Pf</u> 0) $X = X - \overset{\text{open}}{\emptyset}$, $\emptyset = X - \overset{\text{open}}{X}$. ✓

1) Suppose $\{A_\alpha\}_{\alpha \in J}$ are a collection of closed sets.

$$X - \left(\bigcap_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in J} (X - A_\alpha) \quad \text{(by De Morgan's Law)}$$

Since each A_α is ~~open~~ closed, then $X - A_\alpha$ is open.

Since the union of open sets is open, $\bigcup_{\alpha \in J} (X - A_\alpha)$ is open.

Since $X - \left(\bigcap_{\alpha \in J} A_\alpha\right)$ is open, then $\bigcap_{\alpha \in J} A_\alpha$ is closed. $\square$

2) Similar to 1).

Q: How do closed sets relate to the subspace topology?

<u>Prop</u>| Let $A \subset Y \subset X$ where X is a top. space. A is a closed set in Y with the subspace top. iff $A = Y \cap C$ where C is a closed set in X.

<u>Pf</u>| <u>$\Rightarrow$</u>| Supose A is closed in Y.
By def. of closed $Y-A$ is open in Y.
By def of subspace top, $Y-A = Y \cap \mathcal{U}$ for $\mathcal{U}$ open in X. $X - \mathcal{U} = C$ a closed set in X.

<u>Claim</u>| $A = (X-\mathcal{U}) \cap Y$

I leave it to you to check.

<u>$\Leftarrow$</u>| Exercise.