Math 500, Homework 6

Paths, homotopies, and the fundamental group

Due Thursday, 11/30

Reading §51, 52

Exercises (to do on your own)

1. Prove that a group G has a *unique* identity element. Prove that a group element $g \in G$ has a *unique* inverse.

2. Let G and H be groups (with identity elements e_G and e_H). Suppose $\varphi : G \to H$ is a homomorphism. Define $\ker \varphi = \varphi^{-1}(\{e_H\}) = \{g \in G : \varphi(g) = e_H\}$ (the "kernel" of φ).

   (a) Prove that if g^{-1} is the inverse of $g \in G$, then $\varphi(g^{-1})$ is the inverse of $\varphi(g)$.

   (b) Prove that $\ker \varphi$ is a subgroup of G.

   (c) Prove that φ is injective if and only if $\ker \varphi = \{e_H\}$. (In this case, we say φ has "trivial kernel.")

3. Prove that the map $\varphi : G \to H$ defined by $\varphi(g) = e_H$ is a homomorphism. (This is called the "trivial homomorphism".)

4. Verify that the set of integers $\mathbb{Z}$, together with the operation of addition $+$, forms a group. Are the integers a group under the operation of multiplication?

5. Prove that the straight-line homotopy between continuous maps $f, g : X \to \mathbb{R}^n$ is continuous. (Go ahead and assume a version of Lemma 21.4 for addition of vectors and scalar multiplication of vectors.)

Problems (to turn in)

1. Munkres §51, exercise 3 (look at exercise 2 for the definition of $[X, Y]$ and the first definition in §51 for *nulhomotopic*).

2. Munkres §52, exercise 1.

3. Munkres §52, exercise 4.