1.

$$\int \tan^{-1}(x)\,dx$$

use by-parts

$u = \tan^{-1}(x) \qquad v' = 1$

$u' = \dfrac{1}{x^2+1} \qquad v = x$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$$

use u-sub

let $w = x^2+1 \qquad dw = 2x\,dx$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{w}\,dw$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C$$

$$\boxed{\text{So, } \int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C}$$

2. Show $\int_0^{\infty} \frac{\tan^{-1}(x)}{e^x}\,dx$ converges.

Use the comparison theorem for improper int.

Since $0 \le \tan^{-1}(x) \le \frac{\pi}{2}$ for all $x \in [0,\infty)$,

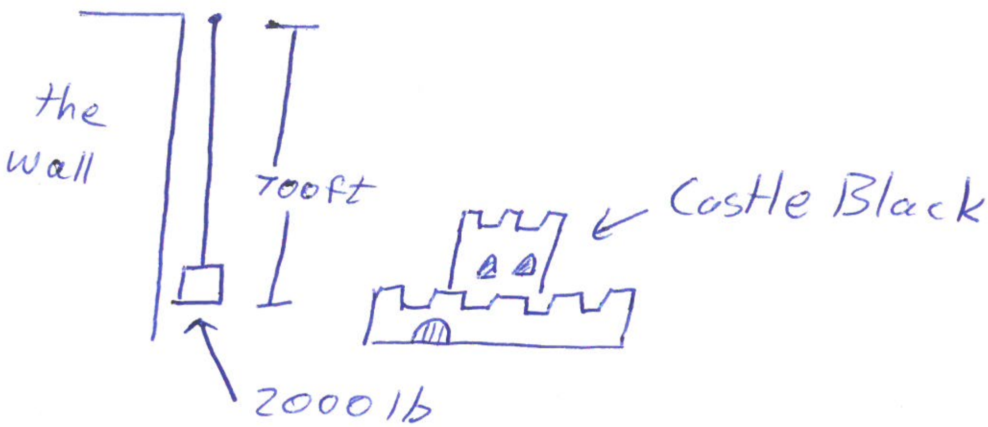
then $0 \le \frac{\tan^{-1}(x)}{e^x} \le \frac{\frac{\pi}{2}}{e^x}$ for all $x \in [0,\infty)$.

Examine
$$\int_0^{\infty} \frac{\frac{\pi}{2}}{e^x}\,dx = \lim_{t\to\infty} \frac{\pi}{2}\int_0^t e^{-x}\,dx$$
$$= \lim_{t\to\infty} \frac{\pi}{2}\left(-e^{-t} + e^{-0}\right)$$
$$= \frac{\pi}{2}(0+1) = \frac{\pi}{2}$$

Since $\int_0^{\infty} \frac{\frac{\pi}{2}}{e^x}\,dx$ converges and $0 \le \frac{\tan^{-1}(x)}{e^x} \le \frac{\frac{\pi}{2}}{e^x}$,

then $\int_0^{\infty} \frac{\tan^{-1}(x)}{e^x}\,dx$ converges by the comparison theorem.

3.



Calculate the total work as the sum of

w_1 = work to hull up chain

+

w_2 = work to hull up ice block

=

w = total work

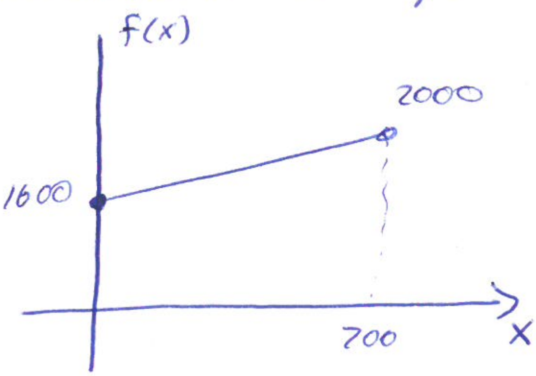
The chain is 700 ft long and weighs 1 lb/ft

$$w_1 = \int_0^{700} 1x\,dx = \frac{x^2}{2}\Big|_0^{700} = \frac{(700)^2}{2} = \frac{490{,}000}{2} = 245{,}000 \text{ ft-lb.}$$

The block weighs 2000 lb initialy and melts at a constant rate untill it weighs $2000 - 50 \cdot 8 = 1600$ lb at the top.

Let $f(x)$ = weight of block distance x from top.

$$f(x) = \frac{4}{7}x + 1600$$



$$w_2 = \int_0^{700} \frac{4}{7}x + 1600\,dx$$

$$= \frac{2}{7}x^2 + 1600x\Big|_0^{700} = 140{,}000 + 1{,}120{,}000 = 1{,}260{,}000 \text{ ft-lb}$$

So, total work $= 245{,}000 + 1{,}260{,}000 = \boxed{1{,}505{,}000 \text{ ft-lb}}$

4. $\cancel{Ra}$ $\sum_{n=1}^{\infty} \frac{(4x-2)^n \ln(n)}{n} = \sum_{n=1}^{\infty} \frac{4^n \ln(n)}{n} (x - \frac{1}{2})^n$

Radius of convergence $= \lim_{n\to\infty} \left| \frac{\frac{4^n \ln(n)}{n}}{\frac{4^{n+1}\ln(n+1)}{n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\ln(n)}{4n\ln(n+1)} \right|$$

$$= \lim_{n\to\infty} \left|\frac{1}{4}\right| \cdot \left|\frac{n+1}{n}\right| \cdot \left|\frac{\ln(n)}{\ln(n+1)}\right|$$

$$= \frac{1}{4} \cdot \left(\lim_{n\to\infty} \left|1+\frac{1}{n}\right|\right) \cdot \left(\lim_{n\to\infty} \left|\frac{\ln(n)}{\ln(n+1)}\right|\right)$$

$$= \frac{1}{4} \cdot (1) \cdot \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right|$$ use L'Hospitals

$$= \frac{1}{4} \cdot (1) \cdot (1)$$

$$= \frac{1}{4}$$

Now, determine conv./div. at $a+R = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and at $a - R = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

$x = a+R = \frac{3}{4}$

$$\sum_{n=1}^{\infty} \frac{4^n \ln(n)}{n} \left(\frac{3}{4} - \frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

use the integral test with $f(x) = \frac{\ln(x)}{x}$, a positive, decreasing, continuous function

$$\int_{\phi}^{\infty} \frac{\ln(x)}{x} = \lim_{t\to\infty} \int_{0=x}^{t=x} u \, du \quad (\text{where } u = \ln(x))$$

$$= \lim_{t\to\infty} \frac{u^2}{2}\Big|_{x=\phi}^{x=t} = \lim_{t\to\infty} \frac{1}{2}(\ln(t))^2 - 0$$

$$= \infty$$

Hence $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ div. by int. test.

4. Continued

$a-R = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$$\sum_{n=1}^{\infty} \frac{4^n \ln(n)}{n} \left(\frac{1}{4}-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

use the alternating series test

$$b_n = \frac{\ln(n)}{n} \qquad b_{n+1} = \frac{\ln(n+1)}{n+1}$$

* Oops * Although it is true that $b_n \geq b_{n+1}$ for n large, it is not easy to show this fact.

After tedius calculation $b_n \geq b_{n+1}$.

Examine $\lim\limits_{n\to\infty} \frac{\ln(n)}{n} \overset{\text{By L.H.}}{=} \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1} = 0.$

Since $\lim\limits_{n\to\infty} b_n = 0$ and $b_n \geq b_{n+1}$, then, by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges.

Hence, the interval of convergence is

$$\left[\frac{1}{4}, \frac{3}{4}\right).$$

5. Show that $\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges conditionally.

Step 1: Show $\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n\ln(n)} \right| = \sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$ diverges.

Let $f(x) = \frac{1}{x\ln(x)}$. $f(x)$ is a decreasing, positive continuous function on $[3, \infty)$.

Examine $\int_3^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_3^{t} \frac{1}{x\ln(x)}\,dx$

use u-sub: Let $u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$\int_3^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_{x=3}^{x=t} \frac{1}{u}\,du = \lim_{t\to\infty} \ln|\ln(x)|\Big|_3^t$$

$$= \lim_{t\to\infty} \ln|\ln(t)| - \ln|\ln(3)| = \infty$$

Hence, by the integral test, since $\int_3^{\infty} \frac{1}{x\ln(x)}\,dx$ diverges, then $\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$ diverges.

Step 2: Show $\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges.

use the alternating series test: $b_n = \frac{1}{n\ln(n)}$, $b_{n+1} = \frac{1}{(n+1)\ln(n+1)}$

Since $n \le n+1$ and $\ln(n) \le \ln(n+1)$, then

$$n\ln(n) \le (n+1)\ln(n+1)$$

$$\frac{1}{n\ln(n)} \ge \frac{1}{(n+1)\ln(n+1)}$$

So $b_n \ge b_{n+1}$. ✓

Examine $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n\ln(n)} = \frac{1}{\infty} = 0$.

~~Hence~~ Since $\lim_{n\to\infty} b_n = 0$ and $b_n \ge b_{n+1}$ for all n, then $\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges by alt. series test.

By steps 1 and 2, $\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges conditionally.

6.

$$f(x) = x e^x \qquad f(0) = 0 \cdot e^0 = 0$$

$$f'(x) = x e^x + e^x$$

$$f''(x) = x e^x + 2e^x$$

$$f'''(x) = x e^x + 3e^x$$

$$f^{(n)}(x) = x e^x + n e^x$$

If $x=0$, $f^{(n)}(0) = 0e^0 + ne^0 = n$

Plug into $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

remember $f(0)=0$.

$$= \sum_{n=0}^{\infty} \frac{n}{n!}(x)^n$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n}$$

7.

A) Given $r = f(\theta)$

$$\left.\begin{aligned} x(\theta) &= f(\theta)\cos(\theta) \\ y(\theta) &= f(\theta)\sin(\theta) \\ \theta &\in (-\infty, \infty) \end{aligned}\right\}$$ This is a parametrization of $r = f(\theta)$.

B) Recall that if y is a differentiable function of x and t and $\cancel{x}$ is a differentiable function of t then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Hence for $r = f(\theta)$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)} = \boxed{\frac{f(\theta)(\cos\theta) + f'(\theta)\sin\theta}{f(\theta)(-\sin\theta) + f'(\theta)\cos\theta}}$$

C) Find the slope of $r = \theta$ at $\theta = \frac{\pi}{2}$

Note $\frac{dr}{d\theta} = 1$

So, $$\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{-\theta\sin\theta + \cos\theta} = \frac{\frac{\pi}{2}\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})}{-\frac{\pi}{2}\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})}$$

$$= \frac{0 + 1}{-\frac{\pi}{2} + 0} = \boxed{-\frac{2}{\pi}}$$

8. We want to find n s.t. $|R_n(x)| < \frac{1}{1000}$

Step 1: Find M s.t. $|f^{(n+1)}(t)| \le M$.

From problem 6, we know $f^{(n+1)}(t) = xe^x + (n+1)e^x$

Since $f^{(n+1)}(t)$ is a strictly increasing function

$$|f^{(n+1)}(t)| \le 1 \cdot e^1 + (n+1)e^1 = e(n+2)$$

Step 2: Plug into Taylor's estimation theorem.

$$|R_n(x)| \le \frac{M\,|x|^{n+1}}{(n+1)!} = \frac{e(n+2)\,|x|^{n+1}}{(n+1)!}$$

Since $x \in [-1, 1]$, then

$$|R_n(x)| \le \frac{e(n+2)\,|1|^{n+1}}{(n+1)!}$$

Step 3: Plug in some values of n on see when

$\frac{e(n+2)}{(n+1)!}$ is less than $\frac{1}{1000}$.

when $n = 7$ $\quad \frac{e(n+2)}{(n+1)!} \approx .00485 > \frac{1}{1000}$

when $n = 8$ $\quad \frac{e(n+2)}{(n+1)!} \approx .000674 < \frac{1}{1000}$

So, $\boxed{n = 8}$