Math 123: Taylor Series

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Outline

Power Series Representations of Functions

We have already shown

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$$

Question: How do we find a power series representation for a general function.

Question: How is the sequence of partial sums of the power series related to the function.

Taylor Series are closely related to approximations

Example: Graph the following functions side-by-side:

- e^x
- 1
- $1 + x$
- $1 + x + \frac{x^2}{2}$
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Core Idea: A power series representation is the LIMIT of successively better polynomial approximations!

Taylor Series

Definition

The **Taylor series** generated by a function f at $x = a$ is

$$\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$$

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Theorem

If $\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ has radius of convergence R, then

$$\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$$

for all x in $(a-R, a+R)$

Tricks to finding Taylor Series

Problem: Find the Taylor series for $f(x) = ln(x+1)$ at $x = 0$.

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Problem: Find the Taylor series for $f(x) = ln(x)$ at $x = 1$.

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Trick: Save yourself time and use the Taylor Series we just found.

Answer: $\Sigma_{k=1}^{\infty}(-1)^{k-1}\frac{(x-1)^k}{k}$

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Problem: Find the first 3 terms of the Taylor series for $f(x) = xsin(3x)$ at $x = 0$.

Trick: Use the fact that you know that Taylor Series for sin(x).

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Problem: Find the first 3 terms of the Taylor series for $f(x) = e^x sin(x)$ at $x = 0$.

Trick: Use the fact that you know that Taylor Series for $sin(x)$ and you know the Taylor Series for e^x.