Math 123: Constant Coefficient 2nd Order Homogeneous Linear D.E.s

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Outline

Types of Differential equations

Definition

A second order **linear** D.E. is of the form

$$y'' + P(x)y' + Q(x)y = R(x)$$

If $R(x) = 0$ we call the D.E. **homogeneous**.

Definition

If $P(x)$ and $Q(x)$ are constants then $y'' + P(x)y' + Q(x)y = R(x)$ is **constant coefficient**.

Solutions to Homogeneous D.E.s

Theorem

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Given a 2nd order homogeneous linear differential equation with **linearly independent** *solutions $f(x)$ and $g(x)$, then the general solution is $y = C_1 f(x) + C_2 g(x)$ where C_1 and C_2 are constants.*

A Motivating Example

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In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.

Case 1: Distinct Roots

If $am^2 + bm + c$ has distinct roots m_1 and m_2, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

.

Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root m_1, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

.

Magic!

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots$$

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$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots$$

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$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots)$$

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$$= cos(\theta) + isin(\theta)$$

Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} cos(\beta x) + c_2 e^{\alpha x} sin(\beta x)$$

.

Auxiliary Equations

Given a linear 2nd order homogeneous **constant-coefficient** differential equation

$$ay'' + by' + cy = 0,$$

the **Auxiliary Equation** is

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The roots of the auxiliary equation determines the general solution.