1. Solve $y'' - 4y' + 4y = 0$ and $y(0) = 1$, $y'(0) = 1$.

Solve the aux. eq.

$$r^2 - 4r + 4 = 0$$
$$(r-2)(r-2) = 0$$
$$r = 2 \quad \text{(a repeated root)}.$$

Hence the general solution is

$$\boxed{y_g = c_1 e^{2x} + c_2 x e^{2x}}$$
$$y_g' = 2c_1 e^{2x} + c_2(2xe^{2x} + e^{2x})$$

Since $y_g(0) = 1$, $1 = c_1 e^0 + c_2 0 \cdot e^0 \Rightarrow \boxed{1 = c_1}$

Since $y_g'(0) = 1$, $1 = 2c_1 e^0 + c_2(2 \cdot 0 \cdot e^0 + e^0)$

$$1 = 2c_1 + c_2$$

Since $c_1 = 1$, $1 = 2 \cdot (1) + c_2$

$$\boxed{c_2 = -1}$$

Thus, $y_g = e^{2x} - x e^{2x}$ is the solution to the IVP.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-2)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 6^n}(x-1)^n = \sum_{n=1}^{\infty} \frac{1}{n^2 3^n}(x-1)^n$

Step 1: Find the radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{\frac{1}{n^2 3^n}}{\frac{1}{(n+1)^2 3^{n+1}}} \right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\right| \cdot \left|\frac{3^{n+1}}{3^n}\right| = 1 \cdot 3 = 3$$

Step 2: Determine convergence or divergence at

$a+R = 1+3 = 4$

$$\sum_{n=1}^{\infty} \frac{(2\cdot 4-2)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test.

$a-R = 1-3 = -2$

$$\sum_{n=1}^{\infty} \frac{(2(-2)-2)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

- Note that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent by p-series.
- Hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutly convergent.
- Since $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutly convergent, then it is convergent.

Hence, the interval of convergence is $[-2, 4]$.

3. Evaluate $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,dx$

use trig sub:

$x = \cos\theta$

$\sqrt{1-x^2} = \sin\theta$

$dx = -\sin\theta\, d\theta$

$$\int \frac{1}{x\sqrt{1-x^2}}\,dx = \int \frac{1}{\cos\theta \cdot \sin\theta} \cdot (-\sin\theta)\,d\theta$$

$$= -\int \sec\theta\, d\theta$$

$$= -\int \frac{\sec\theta}{1} \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

$$= -\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

let $u = \sec\theta + \tan\theta$

$du = \sec\theta\tan\theta + \sec^2\theta\, d\theta$

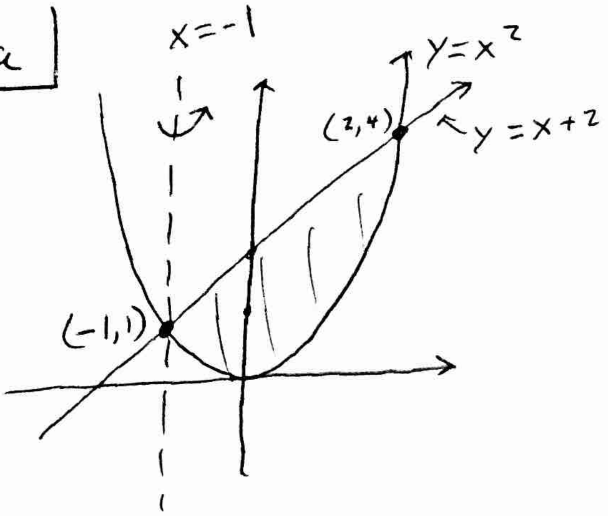
$$= -\int \frac{1}{u}\, du$$

$$= -\ln|u| + c$$

$$= -\ln|\sec\theta + \tan\theta| + c$$

$$= \boxed{-\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C}$$

4a]



$$\int_{-1}^{2} 2\pi (x+2-x^2)(x+1)\,dx$$

4b]

$$\int_{1}^{4} \pi\left((1+\sqrt{y})^2 - (1+(y-2))^2 \right) dy$$

$$+ \int_{0}^{1} \pi\left((1+\sqrt{y})^2 - (1-\sqrt{y})^2 \right) dy$$

5. Examine $\sum_{n=1}^{\infty} \left(\frac{n^2 \tan^{-1}(n)}{\pi n^2+1}\right)^n$

use the root test.

Examine $\lim_{n\to\infty} \left| \left(\frac{n^2 \cdot \tan^{-1}(n)}{\pi n^2+1}\right)^n \right|^{1/n}$

$$= \lim_{n\to\infty} \frac{n^2}{\pi n^2+1} \cdot \tan^{-1}(n)$$

$$= \lim_{n\to\infty} \frac{n^2}{\pi n^2+1} \cdot \lim_{n\to\infty} \tan^{-1}(n)$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{2} = 1/2$$

Since $\lim_{n\to\infty} \left| \left(\frac{n^2 \cdot \tan^{-1}(n)}{\pi n^2+1}\right)^n \right|^{1/n} = 1/2 < 1$, then,

by the root test, $\sum_{n=1}^{\infty} \left(\frac{n^2 \tan^{-1}(n)}{\pi n^2+1}\right)^n$ converges.

6. Solve $y' + \frac{1}{\tan^{-1}(x)\cdot(x^2+1)}\, y = \frac{\ln(x)}{\tan^{-1}(x)}$

Use the integrating factor method.

Step I: Find the integrating factor

$$e^{\int \frac{1}{\tan^{-1}(x)\,(x^2+1)}\,dx}$$

let $u = \tan^{-1}(x)$

$du = \frac{1}{x^2+1}\,dx$

$$= e^{\int \frac{1}{u}\,du}$$

$$= e^{\ln|u|} = |u| = \boxed{\tan^{-1}(x)}$$

Step 2: Multiply both sides by integrating factor and reconize the left as a product rule.

$$\tan^{-1}(x)\, y' + \frac{1}{x^2+1}\, y = \ln(x)$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\cdot y\right) = \ln(x)$$

$$\tan^{-1}(x)\cdot y = \int \ln(x)\,dx$$

use by-parts

$u = \ln(x)$ $\quad v' = 1$

$u' = \frac{1}{x}$ $\quad v = x$

$$\tan^{-1}(x)\cdot y = x\ln(x) - \int 1\,dx = x\ln(x) - x + c$$

$$\boxed{y = \frac{x\ln(x) - x + C}{\tan^{-1}(x)}}$$

7. Bound the error in approximating $f = \ln(1+x)$ by the 21st Taylor polynomial at $x=0$ on the interval $[-1/2, 1/2]$.

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$f''''(x) = \frac{-2\cdot 3}{(1+x)^4}$$

$$f^{(n)}(x) = \frac{(n-1)!\cdot(-1)^{n-1}}{(1+x)^n}$$

Find M s.t. $|f^{(n)}(t)| \leq M$ for all $t \in [-1/2, 1/2]$.

$$|f^{(n)}(t)| = \left|\frac{(n-1)!(-1)^{n-1}}{(1+x)^n}\right| \leq \frac{(n-1)!}{\left(\frac{1}{2}\right)^n} = (n-1)!\,2^n$$

or $|f^{(n+1)}(t)| \leq n!\,2^{n+1}$

Hence, by taylor's estimation theorem

$$|R_{21}(x)| \leq \frac{21!\,2^{22}|x|^{22}}{(22)!} \leq \frac{21!\cdot 2^{22}\left|\frac{1}{2}\right|^{22}}{(22)!} = \boxed{\frac{1}{22}}$$

8. Given $r = \cos\theta$ find all θ s.t. the curve has a horz. tangency.

Plug into $\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

$$\left.\begin{aligned} r &= \cos\theta \\ \frac{dr}{d\theta} &= -\sin\theta \end{aligned}\right\} \text{So, } \frac{dy}{dx} = \frac{-\sin^2\theta + \cos^2\theta}{-\sin\theta\cos\theta - \cos\theta\sin\theta}$$

Solve $0 = -\sin^2\theta + \cos^2\theta$

$\sqrt{\sin^2\theta} = \sqrt{\cos^2\theta}$

$\sin\theta = \pm\cos\theta$

↑ true when $\theta = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$.

Finally, check where the denominator is zero!

$0 = -2\sin\theta\cos\theta$

So $0 = \sin\theta$ or $0 = \cos\theta$

↑ True when $\theta = k\pi; k \in \mathbb{Z}$.

↑ True when $\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

So, the denominator and numerator are never zero at the same time

Thus $\boxed{\theta = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}}$