# MATH 123 PRACTICE MIDTERM 2

NAME (PRINTED): Solutions.

DISCUSSION TIME:

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Please *turn off all electronic devices.* You may use both sides of a 8.5 × 11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**—the grading will be based on your work shown as well as the end result. Remember to put your name at the top of this page. Good luck.

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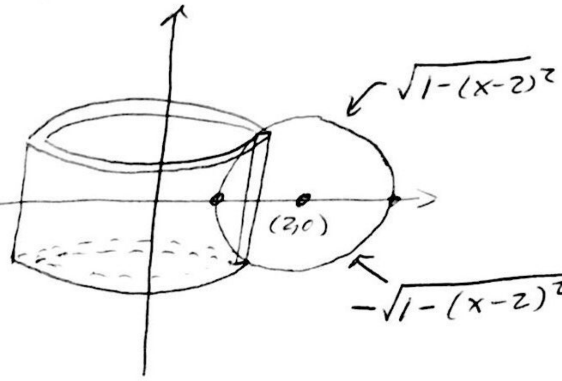
| Problem | Score (out of) |
|---|---|
| 1 | (10) |
| 2 | (10) |
| 3 | (10) |
| 4 | (10) |
| 5 | (10) |
| 6 | (10) |
| 7 | (10) |
| Total | (70) |

**1.** (10 pts) Find the volume of the solid obtained by rotating the region bounded by the circle of radius 1 centered at $(2,0)$ about the $y$-axis.




Step 1: Draw the region

Since the formula for a circle of radius 1 centered at $(2,0)$ is $(x-2)^2+y^2=1$, then

$$y = \pm\sqrt{1-(x-2)^2}$$

Step 2: Use Shell method

radius of shell $= x$

height of shell $= \sqrt{1-(x-2)^2} - \left(-\sqrt{1-(x-2)^2}\right) = 2\sqrt{1-(x-2)^2}$

$$\text{Volume} = \int_1^3 4\pi x\sqrt{1-(x-2)^2}\,dx$$

Do a u-sub
$u = x-2$
$du = dx$
if $x=3$ $u=1$
if $x=1$ $u=-1$

After u-sub

$$\text{Volume} = \int_{-1}^{1} 4\pi(u+2)\sqrt{1-u^2}\,du$$

$$= 4\pi\int_{-1}^{1} u\sqrt{1-u^2}\,du + 4\pi\int_{-1}^{1} 2\sqrt{1-u^2}\,du$$

u-sub: Let $v = 1-u^2$
$dv = -2u\,du$

this is a geometrically known integral (half of a disk of radius 1

$$= 2\int_{?}^{?} -\pi v^{1/2}\,dv \quad + 4\pi\left(2\cdot\tfrac{1}{2}\pi(1)^2\right)$$

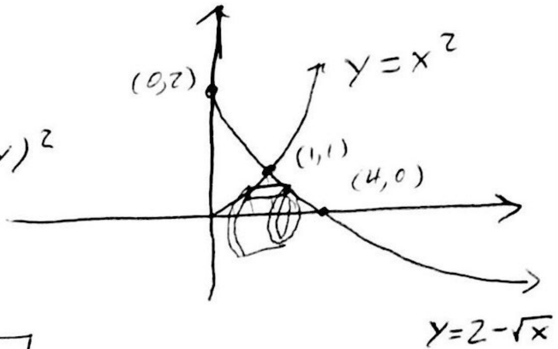
$$= -\tfrac{4}{3}\pi(1-u^2)^{3/2}\Big|_{-1}^{1} \quad + 4\pi^2$$

$$= 0 + 4\pi^2$$

$$= \boxed{4\pi^2}$$

**2.** (10 pts) Write down the integral representing the the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = -\sqrt{x} + 2$ and the x-axis about the x-axis using the:

A) Shell Method Step 1: Draw Region




Solve in terms of y: if $y = x^2$, then $x = \pm\sqrt{y}$

if $y = -\sqrt{x} + 2$, then $x = (2-y)^2$

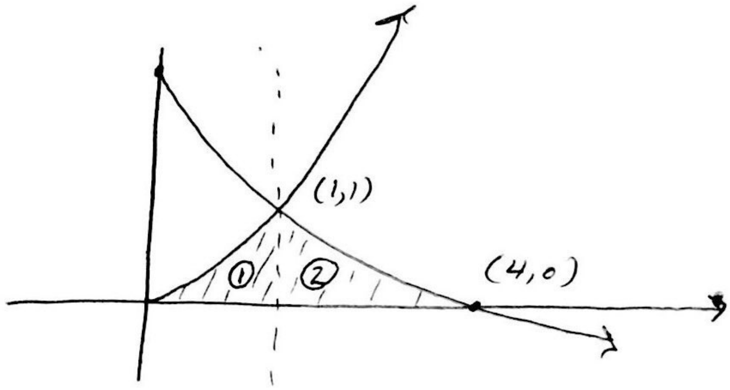
radius of shell = $y$

height of shell = $(2-y)^2 - \sqrt{y}$

$$\boxed{\text{Volume} = \int_0^1 2\pi y\left((2-y)^2 - \sqrt{y}\right)dy}$$

B) Disk Method

Step 1: Draw Region




Must find the vol. of rotating region 1 and add it to vol. of rotating region 2.

Vol. 1: Use disk method

$$\int_0^1 \pi (x^2)^2 dx$$

Vol 2: Use disk method

$$\int_1^4 \pi (2 - \sqrt{x})^2 dx$$

$$\boxed{\text{Total Vol.} = \int_0^1 \pi (x^2)^2 dx + \int_1^4 \pi (2-\sqrt{x})^2 dx}$$

**3.** (10 pts) Solve the following D.E.

$$\frac{dy}{dx} = \frac{(x+1)tan(y)}{(x^2+1)sec^2(y)} = \underbrace{\left(\frac{x+1}{x^2+1}\right)}_{f(x)} \cdot \underbrace{\left(\frac{\tan(y)}{\sec^2(y)}\right)}_{g(y)}$$

Solve: $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

$$\int \frac{\sec^2(y)}{\tan(y)}\,d\cancel{x}y = \int \frac{x+1}{x^2+1}\,dx$$

u-sub
$u = \tan(y)$
$du = \sec^2(y)$

$$\int \frac{1}{u}\,du = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$\ln|\tan(y)| = \int \frac{x}{x^2+1}\,dx + \tan^{-1}(x)$$

↑ u-sub
let $v = x^2+1$
$dv = 2x$

$$\ln|\tan(y)| = \frac{1}{2}\int \frac{1}{v}\,dv + \tan^{-1}(x)$$

$$\ln|\tan(y)| = \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C$$

$$|\tan(y)| = e^{\frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C}$$

$$\tan(y) = \pm e^{c} \cdot e^{\frac{1}{2}\ln|x^2+1| + \tan^{-1}(x)}$$

$(\pm e^c = c)$

$$\boxed{y = \tan^{-1}\left(C \cdot e^{\frac{1}{2}\ln|x^2+1| + \tan^{-1}(x)}\right)}$$

**4.** (10 pts) Solve $y' - 3y = 0$.

$$y' = 3y = \underbrace{3}_{f(x)} \cdot \underbrace{\cancel{y}}_{g(y)}$$

solve

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

$$\int \frac{1}{y}\, dy = \int 3\, dx$$

$$\ln|y| = 3x + c$$

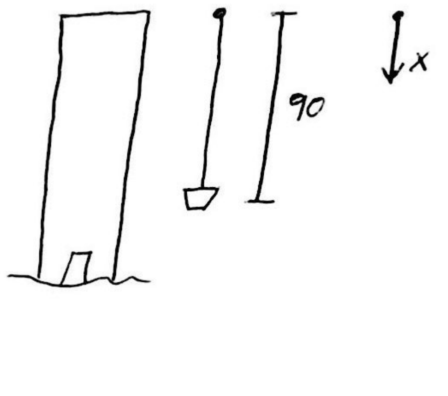
$$|y| = e^{3x+c}$$

$$y = \pm e^{c} e^{3x}$$

$$\boxed{y = C \cdot e^{3x}}$$

**5.** A 60 lb cable is 90 ft long and hangs vertically from the top of a tall building. There is a leaky bucket containing 120 lb of water tied to the end of the cable. If the water leaks out at a constant rate and half the water is gone by the time the cable is lifted to the top of the building, how much work is required to lift the cable and bucket **two-thirds** of the way to the top of the building.

Pic




$x$ = feet from top of building

Total work = $w_1 + w_2 + w_3$

where $w_1$ = work to lift first $\frac{2}{3}$ of cable to top of building

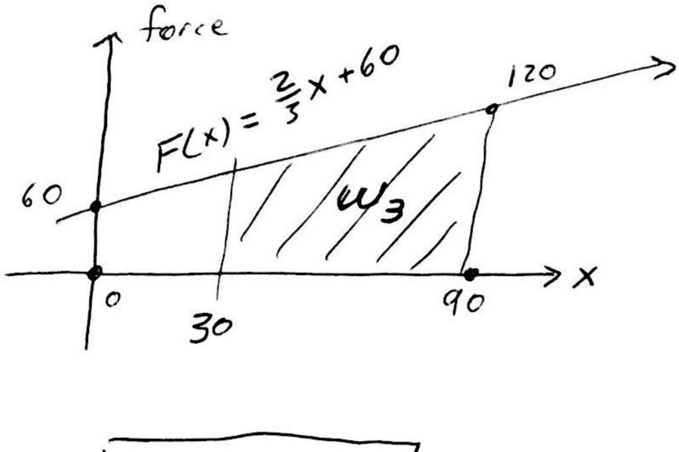
$w_2$ = work to lift last $\frac{1}{3}$ of cable 60ft

$w_2$ = work to lift bucket 60ft.

$$w_1 = \int_0^{60} \frac{60}{90} x \, dx = \int_0^{60} \frac{2}{3} x \, dx = \frac{x^2}{3}\Big|_0^{60} = 1200 \text{ ft-lb.}$$

$$w_2 = (\text{weight of last } 1/3)\cdot(60) \text{ ft-lb} = \left(\frac{60}{90}\cdot 30\right)\cdot(60) = 1200 \text{ ft-lb}$$

$$w_3 = \int_{30}^{90} \frac{2}{3}x + 60 \, dx$$




$$= \frac{x^2}{3} + 60x\Big|_{30}^{90}$$

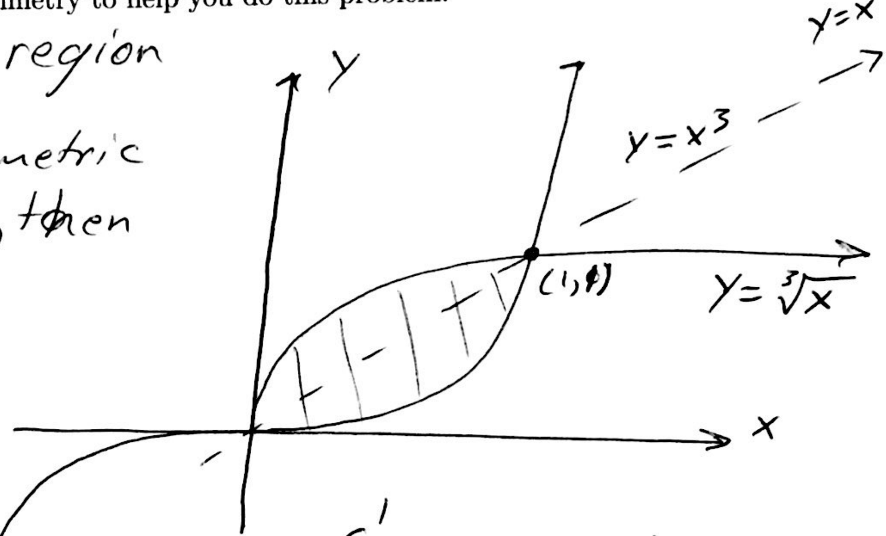
$$= 2700 + 5400 - (300 + 1800)$$

$$= 6000 \text{ ft-lb}$$

Total work = $1200 + 1200 + 6000 = \boxed{8400 \text{ ft-lb}}$

**6.** (10 pts) Find the centroid of the region bounded by $y = x^3$ and $y = x^{\frac{1}{3}}$ between $x = 0$ and $x = 1$. Hint: you may use symmetry to help you do this problem.

Step 1: Draw the region




Since the region is symetric about the line $y = x$, then $\bar{x} = \bar{y}$.

the centroid is

$$\boxed{\left(\frac{16}{35}, \frac{16}{35}\right)}$$

Find $\bar{x} = \dfrac{\int_0^1 x(\sqrt[3]{x} - x^3)\,dx}{\int_0^1 (\sqrt[3]{x} - x^3)\,dx}$

$$\bar{x} = \frac{\int_0^1 x^{4/3} - x^4\,dx}{\int_0^1 x^{1/3} - x^3\,dx}$$

$$\bar{x} = \frac{\frac{3}{7}x^{7/3} - \frac{x^5}{5}\Big|_0^1}{\frac{3}{4}x^{4/3} - \frac{x^4}{4}\Big|_0^1} = \frac{\frac{3}{7} - \frac{1}{5}}{\frac{3}{4} - \frac{1}{4}} \quad \frac{\frac{8}{35}}{\frac{1}{2}} = \frac{16}{35}$$

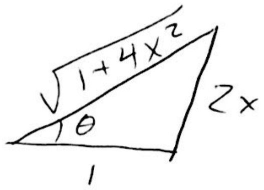
**7.** (10 pts) Find the length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

Plug into $\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$f(x) = x^2$

$f'(x) = 2x$ $\qquad$ length $= \int_0^1 \sqrt{1+4x^2}\,dx$

use trig-sub




$2x = \tan\theta$

$2\,dx = \sec^2\theta\,d\theta$

$\sqrt{1+4x^2} = \sec\theta$

$$\text{length} = \int_?^? \sec\theta \cdot \tfrac{1}{2}\sec^2\theta\,d\theta$$

$$= \tfrac{1}{2}\int_?^? \sec^3\theta\,d\theta$$

By parts $\quad u = \sec\theta \quad v' = \sec^2\theta$

$\qquad\qquad u' = \tan\theta\sec\theta \quad v = \tan\theta$

$$\tfrac{1}{2}\int \sec^3\theta\,d\theta = \tfrac{1}{2}\sec\theta\tan\theta - \tfrac{1}{2}\int \tan^2\theta\sec\theta\,d\theta$$

Note $\quad \tan^2\theta + 1 = \sec^2\theta$

$$\tfrac{1}{2}\int \sec^3\theta\,d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int \sec\theta\,d\theta - \tfrac{1}{2}\int \sec^3\theta\,d\theta$$

$$\int \sec^3\theta\,d\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$\text{So length} = \tfrac{1}{2}(2x)(\sqrt{1+4x^2}) + \tfrac{1}{2}\ln\left|\sqrt{1+4x^2} + 2x\right|\Big|_0^1$$

$$= \boxed{\sqrt{5} + \tfrac{1}{2}\ln(\sqrt{5}+2)}$$