Math 123: Approximate Integration

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Outline

Definition

(**Definite Integral**)If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0(= a), x_1, x_2, ..., x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any **sample points** in these subintervals. Then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)dx = lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

Picking Sample points and Approx. Integrals

$\int_a^b f(x)dx$ is approximated by each of the following

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x$$

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_{n-1})\Delta x$$

$$M_n = f(\frac{1}{2}(x_0 + x_1))\Delta x + f(\frac{1}{2}(x_1 + x_2))\Delta x + ... + f(\frac{1}{2}(x_{n-1} + x_n))\Delta x$$

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Example Find R_4 for $\int_0^1 x^2 dx$ (is this an under estimate or an over estimate?)

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Example Find L_4 for $\int_0^1 x^2 dx$ (is this an under estimate or an over estimate?)

Example Find M_4 for $\int_1^2 e^{x^2} dx$ (is this an under estimate or an over estimate?)

Approximating by Trapezoids

Recall that the area of a trapezoid with parallel sides of length a and b and of height h is

$$A = \frac{1}{2}(a + b)h$$

When we use trapezoids to approximate the area under the curve, we get

$$T_n = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n))$$

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Simpson's Rule

First, find a useful formula for the area under the parabola $Ax^2 + Bx + C$ from $x = -h$ to $x = h$.
We can use this to show that

$$S_n = \frac{1}{3}\Delta x(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots$$

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