Math 123: Taylor Series

Ryan Blair

CSU Long Beach

Thursday November 10, 2016

Outline

Using the geometric series

Exercise: Use

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

to find a power series for $f(x) = \frac{1}{1+x^2}$ and find the interval of convergence.

Derivatives and Integrals of Series

Theorem

If $P(x) = \Sigma_{k=0}^{\infty} c_k (x-a)^k$, *then*

$$P'(x) = \Sigma_{k=1}^{\infty} k c_k (x-a)^{k-1}$$

$$\int P(x)dx = C + \Sigma_{k=0}^{\infty} \frac{c_k}{k+1}(x-a)^{k+1}$$

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Exercise: Find the power series for $f(x) = ln(1+x)$.

Power Series Representations of Functions

We have already shown

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

Question: How do we find a power series representation for a general function.

Question: How is the sequence of partial sums of the power series related to the function.

Taylor Series are closely related to approximations

Example: Graph the following functions side-by-side:

- e^x
- 1
- $1 + x$
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Core Idea: A power series representation is the LIMIT of successively better polynomial approximations!

Taylor Series

Definition

The **Taylor series** generated by a function f at $x = a$ is

$$\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$$

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Exercise: Verify that the Taylor series of $sin(x)$ at $x = 0$ is $\Sigma_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$

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Theorem

If $\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ *has radius of convergence* R, *then*

$$\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$$

for all x *in* $(a-R, a+R)$

Tricks to finding Taylor Series

Problem: Find the Taylor series for $f(x) = ln(x+1)$ at $x = 0$.

Trick: No trick, just substitute into the formula for Taylor series and find the pattern.

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Trick: Save yourself time and use the Taylor Series we just found.

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Trick: Save yourself time and use the Taylor Series we just found.

Answer: $\Sigma_{k=1}^{\infty}(-1)^{k-1}\frac{(x-1)^k}{k}$

Tricks to finding Taylor Series

Problem: Find the first 3 terms of the Taylor series for $f(x) = xsin(3x)$ at $x = 0$.

Trick: Use the fact that you know that Taylor Series for sin(x).

Tricks to finding Taylor Series

Problem: Find the first 3 terms of the Taylor series for $f(x) = e^x sin(x)$ at $x = 0$.

Trick: Use the fact that you know that Taylor Series for $sin(x)$ and you know the Taylor Series for e^x.