Math 600 Day 5: Sard's Theorem

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Outline

Sard's Theorem says that the set of critical values of a smooth map always has measure zero in the receiving space.

We begin with the easiest case, maps of $\mathbb{R}^1 \to \mathbb{R}^1$.

Theorem

Let U be an open set in $\mathbb{R}^1$, and $f : U \to \mathbb{R}^1$ a continuously differentiable map. Let C be the set of critical points of f, that is, $C = \{x \epsilon U : f'(x) = 0\}$. Then $f(C)$ has measure zero.

Proof.

Let I be a closed interval inside U. Define a function $G : I \times I \longrightarrow \mathbb{R}^1$ by

$$G(x,y) = \frac{(f(y) - f(x) - f'(x)(y-x))}{|y-x|}$$

if $x \neq y$

$$= 0$$

if $x = y$.

Since f is continuously differentiable, the map G is continuous, therefore uniformly continuous, and hence $G(x,y)$ can be made arbitrarily close to zero by making x and y sufficiently close to one another.

In particular, given any $\varepsilon > 0$, we can divide the interval I (of length L) into N equal subintervals, each of length $\frac{L}{N}$, so that $|G(x, y)| < \varepsilon$ whenever x and y lie in the same subinterval.

Focus on any one of these subintervals which contains a critical point x of f, so that $f'(x) = 0$. Then for any other point y in that subinterval, we have

$$|f(y) - f(x) - f'(x)(y - x)| = |f(y) - f(x)| < \varepsilon|y - x| \leq \varepsilon\frac{L}{N}.$$

Hence for any two points y_1 and y_2 in this interval, we have

$$|f(y_1) - f(y_2)| \leq 2\varepsilon\frac{L}{N}.$$

Thus the image under f of a subinterval containing a critical point of f is an interval of length $\leq 2\varepsilon\frac{L}{N}$.

There are at most N such subintervals, so the image under f of the critical points which lie in the interval I is contained in a union of intervals of total length $2\varepsilon L$, for any $\varepsilon > 0$, and therefore has measure zero.

Since the open set U is a countable union of such intervals I, it follows that the image under f of all the critical points also has measure zero, completing the proof of the theorem.

Here is the next easiest case.

Theorem

Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ a continuously differentiable map. Let C be the set of critical points of f, that is, $C = \{x \epsilon U : det(f'(x)) = 0\}$. Then $f(C)$ has measure zero.

We will be guided by the proof of Theorem 1, but there will be some changes.

The proof is organized following Milnor's "Topology from the Differentiable Viewpoint," Chapter 3 on pages 16 - 19.

Proof.

Let I be a closed cube inside U of side length L. Define a function $G : I \times I \longrightarrow \mathbb{R}^n$ by

$$G(x,y) = \frac{(f(y) - f(x) - f'(x)(y-x))}{|y-x|}$$

if $x \neq y$

$$= 0$$

if $x = y$.

Since f is continuously differentiable, the map G is continuous, therefore uniformly continuous, and hence $G(x,y)$ can be made arbitrarily close to zero by making x and y sufficiently close to one another.

In particular, given any $\varepsilon > 0$, we can divide the cube I into N^n equal subcubes, each of side length $\frac{L}{N}$, so that $|G(x, y)| < \varepsilon$ whenever x and y lie in the same subcube.

Focus on any one of these subcubes S which contains a critical point x of f, so that $det(f'(x)) = 0$. Then the image of the linear map $f'(x)$ is a proper subspace of $\mathbb{R}^n$, having dimension at most $n - 1$, and is thus contained in some $(n - 1)$-dimensional subspace V.

Then for any other point $y \epsilon S$, the point $f(y)$ lies within $\varepsilon|y - x| \leq \varepsilon n^{\frac{1}{2}} \frac{L}{N}$ of the $(n - 1)$-plane $V + f(x)$.

On the other hand, since f is continuously differentiable, it is certainly Lipschitz, and hence for some constant M, we have

$$|f(y) - f(x)| \leq M|y - x| \leq Mn^{\frac{1}{2}}\frac{L}{N}$$

for any point y in the subcube S containing x.

It follows that the image $f(S)$ lies in a cylinder whose height is less than $2\varepsilon n^{\frac{1}{2}}\frac{L}{N}$, and whose base is an $(n-1)$-dimensional ball of radius $Mn^{\frac{1}{2}}\frac{L}{N}$ lying on the $(n-1)$-plane $V + f(x)$ and centered at $f(x)$.

This cylinder has volume $K\varepsilon(\frac{L}{N})^n$ for some constant K.

There are at most N^n subcubes of the cube I containing critical points of f, so the image under f of all the critical points which lie in the cube I has volume at most

$$K\varepsilon(\frac{L}{N})^n N^n = K\varepsilon L^n.$$

Since this is true for all $\varepsilon > 0$, the set $f(I \cap C)$ of images of critical points of f which lie in the cube I has measure zero.

Since the open set U is a countable union of such cubes I, it follows that $f(C)$, the image under f of all its critical points, also has measure zero, completing the proof.

Here is the next easiest case, requiring some new ideas, and some more differentiability.

Theorem

Let U be an open set in $\mathbb{R}^2$, and $f : U \to \mathbb{R}^1$ a map of class C^3. Let C be the set of critical points of f, that is, $C = \{x \epsilon U : f'(x) = 0\}$. Then $f(C)$ has measure zero.

The added differentiability requirement is essential for the proof.

Proof.

Recalling that C is the set of critical points of f, let $C_i \subset C$ be the set of points $x \epsilon U$ where all partial derivatives of f of order $\leq i$ vanish. Then

$$C = C_1 \supset C_2 \supset C_3.$$

Our proof will be in two steps:

STEP 1. The image $f(C_1 - C_2)$ has measure zero.

STEP 2. The image $f(C_2)$ has measure zero.

This will be sufficient to prove that $f(C)$ has measure zero.

Proof of Step 1.

We must show that the image $f(C_1 - C_2)$ has measure zero.

For each point x_* in $C_1 - C_2$, there is some second derivative $\frac{\partial^2 f}{\partial x_i \partial x_j}$ which is nonzero at x_*. Thus the function $w(x) = \frac{\partial f}{\partial x_j}$ vanishes at x_*, but its derivative $\frac{\partial w}{\partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ does not. Assume $i = 1$.

Now define a map $h : U \to \mathbb{R}^2$ by $h(x) = (w(x), x_2)$. The derivative $h'(x_*)$ is nonsingular, so by the Inverse Function Theorem, h carries some neighborhood V of x_* diffeomorphically onto an open set V' in $\mathbb{R}^2$.

Note that h carries $C_1 \cap V$ into the hyperplane $0 \times \mathbb{R}^1$, since all first derivatives of f vanish at all points of C_1.

Now consider the function $g = f \circ h^{-1} : V' \to \mathbb{R}^1$, and let

$$g^* : (0 \times \mathbb{R}^1) \cap V' \to \mathbb{R}^1$$

denote the restriction of g. By induction, the set of critical values of g^* has measure zero in $\mathbb{R}^1$.

But each point in $h(C_1 \cap V)$ is certainly a critical point of g^*, since all first order derivatives vanish at such points.

Therefore $g^* \circ h(C_1 \cap V) = f(C_1 \cap V)$ has measure zero. Since $C_1 - C_2$ is covered by countably many such sets V, it follows that $f(C_1 - C_2)$ has measure zero, completing Step 1.

Proof of Step 2. We must show that the image $f(C_2)$ has measure zero. Let I be a closed square inside U with edge length L. We will show that $f(C_2 \cap U)$ has measure zero.

By Taylor's Theorem, the compactness of I , and the definition of C_2 , we have that

$$f(x+h) = f(x) + R(x,h)$$

with

$$||R(x,h)|| \leq c||h||^3$$

for $x \epsilon C_2 \cap I$ and $x + h \epsilon I$, where the constant c depends only on f and on I.

It is here that we use the fact that f is of class C^3.

Now subdivide I into N^2 subsquares, each of side length $\frac{L}{N}$.

Let S be a square of this subdivision which contains a point of C_k. Then any point of S can be written as $x + h$, with

$$||h|| \leq \sqrt{2}\frac{L}{N}.$$

From Taylor's Theorem above, it follows that $f(S)$ must lie in an interval of length $2c(\sqrt{2}\frac{L}{N})^3$ centered at f(x) .

There are at most N^2 such subsquares S, hence $f(C_2 \cap I)$ is contained in a union of intervals of total length at most

$$N^2 2c(\sqrt{2}\frac{L}{N})^3 = constant/N.$$

This total length tends to 0 as $N \to \infty$, so $f(C^2 \cap I)$ must have measure zero.

Since U can be covered with countably many such squares I, this shows that $f(C_2)$ has measure zero, completing Step 2, and with it the proof of the theorem.