Math 600 Day 12: Differential Forms

Ryan Blair

University of Pennsylvania

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Induced maps on differential forms.

Now let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map.

Then the derivative of f at each point $p \epsilon \mathbb{R}^n$ is a linear map $f'(p) : \mathbb{R}^m \to \mathbb{R}^n$. We will think of this as a linear map from the tangent space R^m_p to the tangent space $R^n_{f(p)}$, and write it as $f^*(p)$, or simply as f^* . Thus

$$f^* : \mathbb{R}^m_p \to \mathbb{R}^n_{f(p)} \;\; with \;\; f^*(v_p) = (f'(p)(v))_{f(p)}.$$

This linear transformation induces a linear transformation

$$f^* : \Lambda^k(\mathbb{R}^n_{f(p)}) \to \Lambda^k(\mathbb{R}^m_p),$$

which takes a k-form on $\mathbb{R}^n_{f(p)}$ to a k-form on R^m_p.

Now suppose that ω is a differential k-form on $\mathbb{R}^n$. Then we can define a differential k-form $f^*\omega$ on $\mathbb{R}^m$ by

$$(f^*\omega)(p) = f^*(\omega(p)).$$

Remember this means that

$$(f^*\omega)(p)(v_1, ..., v_k) = \omega(f(p))(f(v_1), ..., f(v_k)).$$

If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, the following hold

1. $f^*(dy^i) = \Sigma_{j=1}^{n}(\frac{\partial f^i}{\partial x^j})\ dx^j = \Sigma_{j=1}^{n}(\frac{\partial y^i}{\partial x^j})dx^j$
2. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$
3. $f^*(g \circ \omega) = (g \circ f)f^*(\omega) = f^*(g)f^*(\omega)$
4. $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

Exterior derivative.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function (i.e., 0-form), then df is a differential 1-form. Now we think of "d" as an operator, and extend it so that it takes differential k-forms to differential $k+1$-forms, as follows.

Given the differential k-form

$$\omega = \Sigma_{i_1<...<i_k} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k},$$

we define a differential $k+1$ form $d\omega$, the **differential** or **exterior derivative** of ω, by

$$d\omega = \Sigma_{i_1<...<i_k} d\omega_{i_1,...,i_k} \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$$

$$= \Sigma_{i_1<...<i_k} \Sigma_{r=1}^{n} \left(\frac{\partial \omega_{i_1,...,i_k}}{\partial x^r}\right) dx^r \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

Properties of the Exterior Derivative

1. $d(\omega + \eta) = d\omega + d\eta$
2. If ω and η are differential k- and r-forms, then
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$
3. $d(d\omega) = 0$, or briefly, $d^2 = 0$
4. If $f : \mathbb{R}^m \to \mathbb{R}^n$ and ω is a differential k-form on $\mathbb{R}^n$,
$$f^*(d\omega) = d(f^*\omega).$$

Closed forms and exact forms.

A differential k-form ω is said to be **closed** if $d\omega = 0$, and **exact** if there exists a differential $k-1$ form η such that $\omega = d\eta$.

Every exact form is closed: $\omega = d\eta \Rightarrow d\omega = d(d\eta) = 0$.

If we look only at differential forms defined on all of $\mathbb{R}^n$, then every closed form is exact. But when we look at forms defined on open subsets of $\mathbb{R}^n$, we will find many closed forms which are not exact.

For example, if we look at forms defined on the open set $U = \mathbb{R}^2 - \textit{origin}$, then the 1-form

$$\omega = r(x,y)dx + s(x,y)dy$$
$$= (\frac{-y}{(x^2+y^2)})dx + (\frac{x}{(x^2+y^2)})dy$$

is closed but not exact.

The form ω is closed because $\frac{\partial r}{\partial y} = \frac{\partial s}{\partial x}$.

But it is not exact, because there is no function $f : U \to \mathbb{R}$ with $\frac{\partial f}{\partial x} = r$ and $\frac{\partial f}{\partial y} = s$.

The Poincaré Lemma.

An open set $U \subset \mathbb{R}^2$ is star-shaped with respect to some point $p \epsilon U$ if for each $q \epsilon U$, the line segment from p to q also lies in U.

Theorem

(Poincare Lemma). *If U is a star-shaped open subset of $\mathbb{R}^n$, then every closed form on U is exact.*

Sample proof. We give the argument only for a closed 2-form

$$\omega = r(x, y) \; dx \; + \; s(x, y) \; dy$$

defined on a star-shaped open set U in the plane $\mathbb{R}^2$.

The special case of the Poincaré Lemma that we are proving can be restated as follows.

Theorem

Let r and $s : U \to \mathbb{R}$ be C^1 functions defined on star shaped domain U such that $\frac{\partial r}{\partial y} = \frac{\partial s}{\partial x}$. Then there exists a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial f}{\partial x} = r$ and $\frac{\partial f}{\partial y} = s$.

Proof. First suppose we are given $f(x, y)$ with $f(0,0) = 0$.

Define $g(t) = f(tx, ty)$, and note that, by the chain rule,

$$g'(t) = (\frac{\partial f}{\partial x})(tx, ty)x + (\frac{\partial f}{\partial y})(tx, ty)y.$$

Then

$$f(x, y) = g(1) = \int_0^1 g'(t)dt$$

$$= \int_0^1 [(\frac{\partial f}{\partial x})(tx, ty)x + (\frac{\partial f}{\partial y})(tx, ty)y]dt.$$

Therefore, to find a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = r$ and $\frac{\partial f}{\partial y} = s$, we should **define** f by

$$f(x, y) = \int_0^1 [r(tx, ty)x + s(tx, ty)y]dt,$$

and aim to show that $\frac{\partial f}{\partial x} = r$ and $\frac{\partial f}{\partial y} = s$.

Exercise. Let U and V be open subsets of $\mathbb{R}^n$ with U star-shaped. Suppose there is a diffeomorphism $f : U \to V$. Show that every closed form on V is exact.

De Rham Cohomology.

If U is an open subset of $\mathbb{R}^n$, we let

$\Omega^k(U)=$ vector space of smooth k-forms on U.

Since $d^2 = 0$, the image of $d : \Omega^{k-1}(U) \to \Omega^k(U)$ is a subspace of the kernel of $d : \Omega^k(U) \to \Omega^{k+1}(U)$.

The corresponding quotient space,

$$H^k_{DeR}(U) = \frac{ker(d : \Omega^k(U) \to \Omega^{k+1}(U))}{im(d : \Omega^{k-1}(U) \to \Omega^k(U))} = \frac{\{closed \ \ k - forms \ \ on \ \ U\}}{\{exact \ \ k - forms \ \ on \ \ U\}}$$

is called the **kth DeRham cohomology group** of U (actually a real vector space).

Theorem Let $f : U \to V$ be a diffeomorphism between open sets of $\mathbb{R}^n$. Show that for each $k = 0, 1, ..., n$, f induces an isomorphism

$$f^* : H^k_{DeR}(V) \to H^k_{DeR}(U).$$