STAT 482 SECOND MIDTERM EXAM SOLUTION

Problem 1. People come to a library to check out books according to a Poisson process with a rate of four per hour. Each person, independently of others, will check out one, two, three, or four books with respective probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, and $\frac{1}{8}$.

(a) Give the expression for the total number of books checked out within t hours. Give the name of the process and specify all parameters.

Let $X(t)$ denote the total number of books checked out by time t, and let Y_i be the number of books checked out by the ith individual. We can write $X(t) = \sum_{i=1}^{N(t)} Y_i$ where $N(t) \sim Poisson(4t)$ and is independent of $Y_i, i = 1, \dots, N(t)$, which have the pmf $p(1) = \frac{1}{2}, p(2) = \frac{1}{4}, p(3) = p(4) = \frac{1}{8}$.

(b) Find the mean and standard deviation of the total number of books checked out within six hours.

$$\boldsymbol{E\big(X(6)\big) = \lambda t E(Y_1) = (4)(6)\left((1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) + (3)\left(\frac{1}{8}\right) + (4)\left(\frac{1}{8}\right)\right) = 45,}$$

$$\boldsymbol{\sqrt{Var(X(t))} = \sqrt{\lambda t E\left(Y_1^2\right)} = \sqrt{(4)(6)\left((1)^2\left(\frac{1}{2}\right) + (2)^2\left(\frac{1}{4}\right) + (3)^2\left(\frac{1}{8}\right) + (4)^2\left(\frac{1}{8}\right)\right)}}$$

$$\boldsymbol{= \sqrt{111} = 10.53565.}$$

Problem 2. People arrive at a nightclub according to a Poisson process with a rate of five per hour between 7 PM and 10 PM, 10 per hour between 10 PM and 12 AM, and 4 per hour between 12 AM and 4 AM.

(a) Describe the random process in words. What is the name of this process? Write down the intensity rate function and compute the mean-value function for this process.

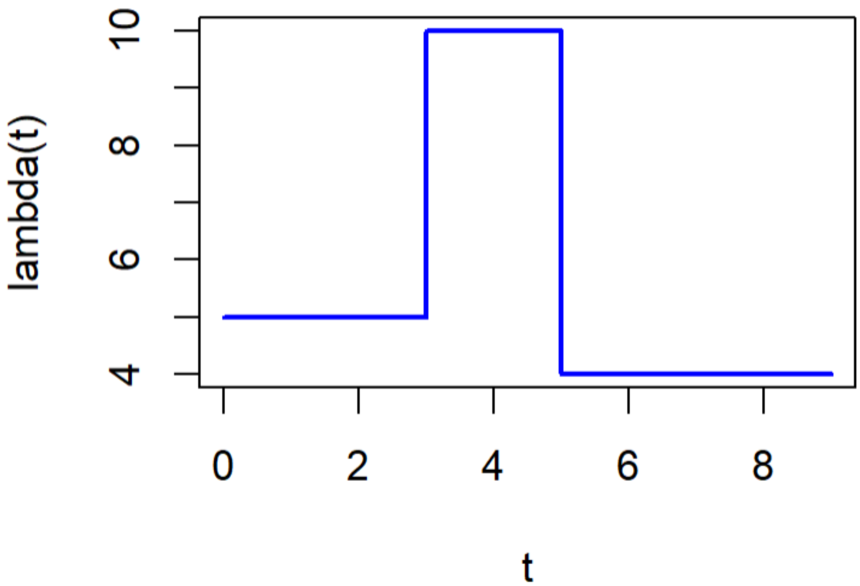
The random process $\{N(t), t \geq 0\}$ counts the number of arrivals by time t. It is a nonhomogeneous Poisson process with the intensity rate

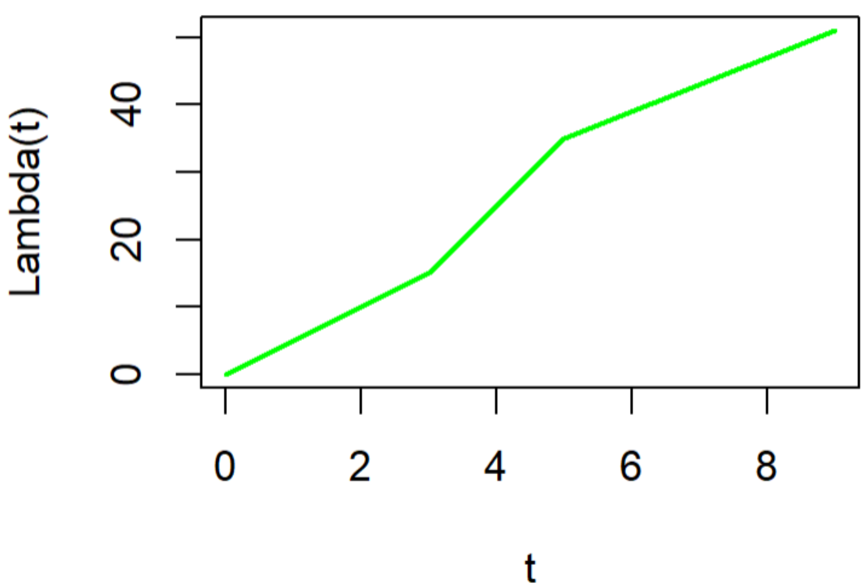
$$\boldsymbol{\lambda(t) = \begin{cases} 5, & 0 \leq t \leq 3, \\ 10, & 3 < t \leq 5, \\ 4, & 5 < t \leq 9. \end{cases}}$$

The integrated intensity rate function (or the mean-value function) is found as

$$\Lambda(t) = \int_0^t \lambda(s)ds = \begin{cases} \int_0^t 5ds = 5t, & 0 \le t \le 3, \\ 15 + \int_3^t 10ds = 15 + 10(t-3) = 10t - 15, & 3 < t \le 5, \\ 35 + \int_5^t 4ds = 35 + 4(t-5) = 4t + 15, & 5 < t \le 9. \end{cases}$$

(b) Draw two graphs, one underneath the other, of the intensity rate function and the mean-value function.





(c) What is the average number of customers who arrive between 9 PM and 2 AM?

$$\boldsymbol{E(N(7)-N(2)) = \Lambda(7)-\Lambda(2) = (4)(7)+15-(5)(2) = 33.}$$

(d) What is the probability that exactly 22 customers arrive between 9 PM and 2 AM?

$$\begin{aligned}\boldsymbol{P(N(7)-N(2)=22)} &= \boldsymbol{\frac{(\Lambda(7)-\Lambda(2))^{22}}{22!}e^{-(\Lambda(7)-\Lambda(2))}} \\ &= \boldsymbol{\frac{((4)(7)+15-(5)(2))^{22}}{22!}e^{-((4)(7)+15-(5)(2))} = \frac{(33)^{22}}{22!}e^{-33}} \\ &= \boldsymbol{0.010588.}\end{aligned}$$

Problem 3. Pilots contact the air-traffic controller (ATC) on the radio according to a Poisson process with a rate Λ per hour, where Λ is uniformly distributed between 5 and 12.

(a) Describe the random process in words. What is the name of this process? Specify all the underlying distributions and their parameters.

Let $N(t)$ be the number of pilots who contact the ATC by time t. Then $\{N(t), t \geq 0\}$ is a conditional Poisson process such that $N(t)|\,\Lambda \sim Poisson\,(\Lambda t)$ where $\Lambda \sim Unif(5,12)$.

(b) Given that the observed value of Λ is 7.8, find the conditional probability that 13 pilots contact the ATC within two hours.

$$\boldsymbol{P(N(2)=13 \mid \Lambda = 7.8) = \frac{((7.8)(2))^{13}}{13!}e^{-(7.8)(2)} = 0.0874.}$$

(c) Suppose we want to compute the marginal distribution that 13 pilots contact the ATC within two hours. Write down the expression that needs to be evaluated in order to calculate the answer. Simplify it to the best of your ability, but don't compute numerically.

$$\begin{aligned}\boldsymbol{P(N(2)=13)} &= \boldsymbol{\int_0^\infty P(N(2)=13 \mid \Lambda=\lambda)\, f_\Lambda(\lambda) d\lambda = \int_5^{12} \frac{(2\lambda)^{13}}{13!}e^{-2\lambda}\cdot\frac{1}{7}d\lambda} \\ &= \boldsymbol{\frac{2^{13}}{(13!)(7)}\int_5^{12} \lambda^{13}e^{-2\lambda}\,d\lambda\,.}\end{aligned}$$

(d) What is the average number of pilots who contact the ATC within two hours? What is the variance?

$$\boldsymbol{EN(2) = tE\Lambda = (2)\frac{5+12}{2} = 17,}$$

$$\boldsymbol{Var\big(N(2)\big) = t^2Var(\Lambda) + tE\Lambda = 2^2\frac{(12-5)^2}{12} + 17 = \frac{49}{3} + 17 = \frac{100}{3} = 33.33.}$$