3.2. The Analysis of Variance.

Example. A textile company weaves a certain fabric on three looms. It is suspected that different looms produce fabric of different strength. To test the claim, a quality control engineer randomly chooses four samples of fabrics for every loom. The observed data are textile strength in pounds per square inch.

Looms	Strength			
1	88	93	90	89
2	91	89	92	94
3	97	96	94	93

Definition. In a one-way analysis of variance (one-way ANOVA), a single factor (treatment) with several levels is considered.
In our example, the treatments are the looms. There are three treatments (three levels of the factor).
In a one-way problem, the data are presented by the following table

Treatments	Observations				**Totals**	**Averages**
1	y_{11}	y_{12}	$\cdots$	y_{1n}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n}	$y_{2.}$	$\bar{y}_{2.}$
.	.	.	$\cdots$	.	.	.
.	.	.	$\cdots$	.	.	.
.	.	.	$\cdots$	.	.	.
a	y_{a1}	y_{a2}	$\cdots$	y_{an}	$y_{a.}$	$\bar{y}_{a.}$
					$y_{..}$	$\bar{y}_{..}$

Definition. The statistical model for the observations is $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, where y_{ij} is the jth observation for the ith treatment (the ijth observation), $i = 1, \dots, a, \ \ j = 1, \dots, n$. The parameter μ is the overall mean, τ_i is the ith treatment effect, and ε_{ij} is a random error. It is assumed that ε_{ij}'s are i.i.d. $N(0, \sigma^2)$. This is a completely additive model (there is no interaction).
ANOVA hypotheses. We are interested in testing for zero treatment effect, that is, we would like to test $H_0 : \tau_1 = \tau_2 = \cdots = \tau_a = 0$ versus $H_1 : \tau_i \neq 0$ for some i.

Definition. Alternatively, the statistical model is given by $y_{ij} = \mu_i + \varepsilon_{ij}$, where μ_i is the mean of the ith factor level ($\mu_i = \mu + \tau_i$). The ANOVA hypotheses for this model are $H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$ versus $H_1 : \mu_i \neq \mu_j$ for some i and j.
Notice the relations $\mu = \sum_{i=1}^{a} \mu_i / a$ and $\sum_{i=1}^{a} \tau_i = 0$.

Derivation of the test statistic.

Definition. A fixed effects model is the model in which levels of the factor are chosen by the experimenter. In our example, the three looms are the only looms at the factory, so it is a fixed effects model.
Definition. A random effects model is the model in which the a treatments are a random sample from a larger population of treatments. For example, a textile factory has one hundred looms. A quality control engineer randomly chooses three looms, and obtains four random samples of fabrics for every loom. This is an example of a random effects model.

We derive the test statistic for the fixed effects model.
Notation.

$$y_{i.} = \sum_{j=1}^{n} y_{ij}, \quad \bar{y}_{i.} = \frac{y_{i.}}{n}, \quad i = 1, \dots, a,$$

$$y_{..} = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}, \quad \bar{y}_{..} = \frac{y_{..}}{N}, \quad N = na.$$

3.3.1. Decomposition of the Total Sum of Squares.

Definition. The total corrected sum of squares is $SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2$. It represents the total variability in the data.
It can be partitioned into component parts (whence, the name "Analysis of Variance"). It can be written as

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{a} \sum_{j=1}^{n} [(\bar{y}_{i.} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.})]^2$$

$$= n \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.})^2 + 2 \sum_{i=1}^{a} \sum_{j=1}^{n} (\bar{y}_{i.} - \bar{y}_{..})(y_{ij} - \bar{y}_{i.}).$$

The cross product is equal to zero:

$$\sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.}) = y_{i.} - \bar{y}_{i.} = y_{i.} - n(y_{i.}/n) = 0.$$

Thus, $SS_T = SS_{tr} + SS_E$ where $SS_{tr} = n \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2$ is the sum of squares due to treatments (between treatments), and $SS_E = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.})^2$ is the sum of squares due to error (within treatments).

3.3.2. Fixed Effects Model: Statistical Analysis.

Recall that $SS_T = SS_{tr} + SS_E$ where $SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2$, $SS_{tr} = n \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2$, $SS_E = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.})^2$.

The quantities $MS_{tr} = \frac{SS_{tr}}{a-1}$ and $MS_E = \frac{SS_E}{N-a}$ are called mean squares. It can be shown that

$$\mathbb{E}(MS_{tr}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\tau_i^2}{a-1} \text{ and } \mathbb{E}(MS_E) = \sigma^2.$$

Under H_0, both MS_E and MS_{tr} are estimates of σ^2. Moreover, it can be shown (Cochran's Theorem, page 69) that, under H_0, SS_E/σ^2 and SS_{tr}/σ^2 are independent chi-square random variables with $N-a$ and $a-1$ degrees of freedom. Therefore, if the null hypothesis is true, the ratio

$$F = \frac{SS_{tr}/(a-1)}{SS_E/(N-a)} = \frac{MS_{tr}}{MS_E}$$

has an F distribution with $a-1$ and $N-a$ degrees of freedom. Thus, one would reject the null, if $F > F_{\alpha, a-1, N-a}$.

Computational Formulas.

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N},$$

$$SS_{tr} = \sum_{i=1}^{a}\frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N},$$

$$SS_E = SS_T - SS_{tr}.$$

The one-way ANOVA table looks like this

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Between treatments	SS_{tr}	$a-1$	MS_{tr}	$F = \frac{MS_{tr}}{MS_E}$
Error(within treatments)	SS_E	$N-a$	MS_E	
Total	SS_T	$N-1$		

Example. In the example with looms, test whether the looms produce fabric of different strength. Use $\alpha = 0.05$.
Solution: We want to test $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$ against $H_1 : \tau_i \neq 0$ for some $i = 1, 2, 3$.

Looms	Strength				Totals
1	88	93	90	89	360
2	91	89	92	94	366
3	97	96	94	93	380
					1106

$a = 3,\ n = 4,\ N = 12,\ SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N} = 102026 - 101936.33 = 89.67,\quad SS_{tr} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N} = 101989 - 101936.33 = 52.67,\quad SS_E = SS_T - SS_{tr} = 37,\ MS_{tr} = \frac{SS_{tr}}{a-1} = \frac{52.67}{2} = 26.34,\ MS_E = \frac{SS_E}{N-a} = \frac{37}{9} = 4.11,\ F = \frac{MS_{tr}}{MS_E} = \frac{26.34}{4.11} = 6.41$. The ANOVA table is

Source	SS	df	MS	F
Looms	52.67	2	26.34	6.41
Error	37	9	4.11	
Total	89.67	11		

$F_{0.05,\,2,\,9} = 4.26 \ \Rightarrow$ reject H_0 and conclude that the fabrics differ in strength.

3.3.3. Fixed Effects Model: Estimation of Model Parameters.

Proposition. If $X_i \sim N(\mu,\ \sigma^2)$ where $i = 1, \dots, n$, then the MLE of μ is $\hat{\mu} = \bar{X} = (X_1 + \cdots + X_n)/n$.

The observations in a one-factor model satisfy $y_{ij} = \mu_i + \varepsilon_{ij} \sim N(\mu_i,\ \sigma^2)$ for all $j = 1, \dots, n$ and some fixed $i = 1, \dots, a$. Therefore, the MLE of μ_i is $\hat{\mu}_i = (y_{i1} + \cdots + y_{in})/n = \bar{y}_{i.}$. Also, since the overall mean $\mu = \sum \mu_i/a$, its MLE is $\hat{\mu} = \sum \hat{\mu}_i/a = (\bar{y}_{1.} + \cdots + \bar{y}_{a.})/a = (y_{1.}/n + \cdots + y_{a.}/n)/a = (y_{1.} + \cdots + y_{a.})/N = \bar{y}_{..}$.
Writing the model as $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, we see that the MLEs of the model parameters are $\hat{\mu} = \bar{y}_{..}$ and $\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$.

In the example with looms, $\hat{\mu} = 1106/12 = 92.17$, $\hat{\mu}_1 = 360/4 = 90$, $\hat{\mu}_2 = 366/4 = 91.5$, $\hat{\mu}_3 = 380/4 = 95$, $\hat{\tau}_1 = 90 - 92.17 = -2.17$, $\hat{\tau}_2 = 91.5 - 92.17 = -0.67$, $\hat{\tau}_3 = 95 - 92.17 = 2.83$.

Confidence Intervals.

The distribution of $\bar{y}_{i.}$ is $N(\mu_i, \sigma^2/n)$. Use MS_E as an estimate for σ^2. It can be shown that $\frac{\bar{y}_{i.} - \mu_i}{\sqrt{MS_E/n}} \sim t_{N-a}$. Therefore, a $100(1-\alpha)\%$ confidence interval for the ith treatment mean μ_i is $\bar{y}_{i.} \pm t_{\alpha/2,\,N-a}\sqrt{MS_E/n}$. A $100(1-\alpha)\%$ confidence interval for the difference in two treatment means

$\mu_i - \mu_j$ is $\bar{y}_{i.} - \bar{y}_{j.} \pm t_{\alpha/2,\,N-a}\sqrt{2MS_E/n}$.

In our example, the 95% confidence intervals for the treatment means and the difference in treatment means are computed as follows:

$$\bar{y}_{1.} = 90,\ \bar{y}_{2.} = 91.5,\ \bar{y}_{3.} = 95,\ MS_E = 4.11,\ N = 12,\ a = 3,\ t_{.025,\,9} = 2.262,$$

$$n = 4,\ \ 87.71 \le \mu_1 \le 92.29,\ \ 89.21 \le \mu_2 \le 93.79,\ \ 92.71 \le \mu_3 \le 97.29,$$

$$-1.74 \le \mu_2 - \mu_1 \le 4.74,\ \ 1.76 \le \mu_3 - \mu_1 \le 8.24,\ \ 0.26 \le \mu_3 - \mu_2 \le 6.74.$$

Notice that the confidence interval for the difference between μ_2 and μ_1 includes zero, but the other two intervals don't. It means that there is no statistical difference between looms one and two, but there is difference between looms one and three, and two and three.

Example. A manufacturer of television sets is interested in the effect on tube conductivity of four different types of coating for color picture tubes. The following conductivity data are obtained:

Coating Type	Conductivity				Total
1	143	141	150	146	580
2	152	149	137	143	581
3	134	136	132	129	531
4	129	127	132	129	517
					2209

(a) Write down the math model for this experiment.
Solution: $y_{ij} = \mu_i + \varepsilon_{ij},\ i = 1, \dots, 4,\ j = 1, \dots, 4,\ \varepsilon \overset{iid}{\sim} N(0,\ \sigma^2),\ \hat{\mu}_1 = 145,\ \hat{\mu}_2 = 145.25,\ \hat{\mu}_3 = 132.75,\ \hat{\mu}_4 = 129.25,\ SS_T = 1040.94,\ SS_{tr} = 822.69,\ SS_E = 218.25,\ MS_E = 18.19,\ \hat{\sigma} = \sqrt{MS_E} = 4.26.$
Or, alternatively, $y_{ij} = \mu + \tau_i + \varepsilon_{ij},\ i = 1, \dots, 4,\ j = 1, \dots, 4,\ \varepsilon \overset{iid}{\sim} N(0,\ \sigma^2),\ \hat{\sigma} = 4.26, \hat{\mu} = 138.06,\ \hat{\tau}_1 = 6.94,\ \hat{\tau}_2 = 7.19,\ \hat{\tau}_3 = -5.31,\ \hat{\tau}_4 = -8.81.$

(b) Construct the 99% confidence intervals for the treatment means and for the differences in treatment means.
Solution: $t_{0.005,\,12} = 3.055,\ 138.49 \le \mu_1 \le 151.51,\ 138.74 \le \mu_2 \le 151.76,\ 126.24 \le \mu_3 \le 139.26,\ 122.74 \le \mu_4 \le 135.76,\ -8.98 \le \mu_2 - \mu_1 \le 9.48,\ 3.02 \le \mu_1 - \mu_3 \le 21.48,\ -5.73 \le \mu_3 - \mu_4 \le 12.73.$

3.5.7. Comparing Pairs of Treatment Means.

To compare all pairs of treatment means in a one-way ANOVA model, we test simultaneously $\binom{a}{2}$ pairs of hypotheses $H_0 : \mu_i = \mu_j$ for all $i \ne j$ against $H_1 : \mu_i \ne \mu_j$ for some $i \ne j$.

We study three tests: Tukey's test, the Fisher Least Significant Difference

(LSD) method, and Duncan's Multiple Range test.

Tukey's Honestly Significant Difference (HSD) Test.

The test declares two means significantly different if the absolute value of the difference of the respective sample treatment means exceeds
$T_\alpha = q_\alpha(a, N-a)\sqrt{MS_E/n}$ where the critical value $q_\alpha(a, N-a)$ comes from table VIII on pages 656-657.
Example. In the example with looms, we would like to test simultaneously three pairs of hypotheses $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$, $H_0 : \mu_2 = \mu_3$ against $H_1 : \mu_2 \neq \mu_3$, and $H_0 : \mu_1 = \mu_3$ against $H_1 : \mu_1 \neq \mu_3$.
The three sample treatment means are $\bar{y}_{1.} = 90$, $\bar{y}_{2.} = 91.5, \bar{y}_{3.} = 95$. The differences in the means are
$\bar{y}_{1.} - \bar{y}_{2.} = -1.5$
$\bar{y}_{2.} - \bar{y}_{3.} = -3.5$
$\bar{y}_{1.} - \bar{y}_{3.} = -5.$
The absolute values of these quantities should be compared to
$T_{0.05} = q_{0.05}(3, 9)\sqrt{MS_E/n} = 3.95\sqrt{4.11/4} = 4.00$. The conclusion is that there is a significant difference between μ_1 and μ_3 but the differences between μ_1 and μ_2, and μ_2 and μ_3 are insignificant. This can be presented graphically as

$\bar{y}_{1.}$ $\bar{y}_{2.}$ $\bar{y}_{3.}$

The Fisher Least Significant Difference (LSD) Method.

The means μ_i and μ_j are significantly different if $|\bar{y}_{i.} - \bar{y}_{j.}| > t_{\alpha/2, N-a}\sqrt{2MS_E/n}$.

Example. In our example, $t_{0.025, 9}\sqrt{2MS_E/n} = 2.262\sqrt{4.11/2} = 3.24$. The conclusion is that μ_1 and μ_2 are not different, but μ_1 and μ_3 and μ_2 and μ_3 are different.

$\bar{y}_{1.}$ $\bar{y}_{2.}$ $\bar{y}_{3.}$

Duncan's Multiple Range Test.

Step 1. Arrange the sample means in increasing order.
Step 2. Compute $a-1$ quantities $R_p = r_\alpha(p, N-a)\sqrt{MS_E/n}$, $p = 2, \ldots a$, where the significant ranges $r_\alpha(p, N-a)$ can be found in table VII on page 655.
Step 3. Compare the difference between the largest and the smallest means to R_a, between the largest and the second-smallest to R_{a-1}, and so on, until all means are compared to the largest. Then, the difference between the second-largest and the smallest means are compared to R_{a-1}, etc. Until all $\binom{a}{2}$ pairs of means has been considered.

Example. $R_2 = r_{0.05}(2,\,9)\sqrt{MS_E/n} = 3.2\sqrt{4.11/4} = 3.24$,
$R_3 = r_{0.05}(3,\,9)\sqrt{MS_E/n} = 3.34\sqrt{4.11/4} = 3.39$.
$\bar{y}_{3.} - \bar{y}_{1.} = 5 > 3.39(R_3)$
$\bar{y}_{3.} - \bar{y}_{2.} = 3.5 > 3.24(R_2)$
$\bar{y}_{2.} - \bar{y}_{1.} = 1.5 < 3.24(R_2)$
The conclusion is that μ_1 and μ_3, as well as μ_2 and μ_3 differ significantly, but μ_1 and μ_2 do not.

$\underline{\bar{y}_{1.} \qquad \bar{y}_{2.}} \qquad \underline{\bar{y}_{3.}}$

3.5.4. Contrasts.

Definition. A contrast is a linear combination of the treatment means of the form $\Gamma = \sum_{i=1}^{a} c_i\,\mu_i$ where the coefficients $c_1, \dots, c_a$ sum to zero, that is, $\sum_{i=1}^{a} c_i = 0$.
The goal is to test $H_0 : \sum_{i=1}^{a} c_i\mu_i = 0$ against $H_1 : \sum_{i=1}^{a} c_i\mu_i \neq 0$.
We estimate the contrast by $\hat{\Gamma} = \sum_{i=1}^{a} c_i\bar{y}_{i.}$. The distribution of $\hat{\Gamma}$ is $N(\sum_{i=1}^{a} c_i\mu_i,\, \sum_{i=1}^{a} c_i^2\sigma^2/n)$. Therefore, under the null hypothesis, the variable

$$\frac{\sum_{i=1}^{a} c_i\bar{y}_{i.}}{\sqrt{\sum_{i=1}^{a} c_i^2\sigma^2/n}}$$

has a $N(0,1)$ distribution. Thus, the test statistic

$$t = \frac{\sum_{i=1}^{a} c_i\, y_{i.}}{\sqrt{nMS_E \sum_{i=1}^{a} c_i^2}}.$$

has the t-distribution with $N - a$ degrees of freedom. The null hypothesis should be rejected if $|t| > t_{\alpha/2,\,N-a}$.
Or, equivalently, $F = t^2 = \frac{(\sum_{i=1}^{a} c_i\, y_{i.})^2}{nMS_E \sum_{i=1}^{a} c_i^2}$ has F distribution with 1 and $N - a$ degrees of freedom. The null should be rejected if $F > F_{\alpha,\,1,\,N-a}$.

Example. In the example with looms, suppose we would like to test $H_0 : \mu_2 = (\mu_1 + \mu_3)/2$ against $H_1 : \mu_2 \neq (\mu_1 + \mu_3)/2$. The contrast $\Gamma = \mu_1 - 2\mu_2 + \mu_3$.
Notice that, for example, testing $H_0 : \mu_3 = 2\mu_1$ cannot be expressed in terms of a contrast since $\Gamma = 2\mu_1 - \mu_3$ is not a contrast.
The test statistic is

$$t = \frac{360 - (2)(366) + 380}{\sqrt{(4)(4.11)(1+4+1)}} = 0.81.$$

The critical value $t_{0.025,\,9} = 2.262$. We accept the null and conclude that the mean strength for loom 2 equals the average of the means for loom 1 and 3.

3.5.5. Orthogonal Contrasts.

Definition. Two contrasts with coefficients $\{c_i\}$ and $\{d_i\}$ are orthogonal if $\sum_{i=1}^{a} c_i\, d_i = 0$.
Example. $\Gamma_1 = \mu_1 - 2\mu_2 + \mu_3$ and $\Gamma_2 = \mu_1 - \mu_3$ are orthogonal since $(1)(1) + (-2)(0) + (1)(-1) = 0$.

For a treatments the set of $a-1$ orthogonal contrasts partition the sum of squares due to treatments into $a-1$ independent single-degree-of-freedom components. Denote by C a contrast in terms of the treatment totals, that is, $C = \sum_{i=1}^{a} c_i y_{i.}$. Then the contrast sum of squares is

$$SS_C = \frac{(\sum_{i=1}^{a} c_i y_{i.})^2}{n\sum_{i=1}^{a} c_i^2}.$$

It can be shown by noticing that $F = t^2 = \frac{(\sum_{i=1}^{a} c_i\, y_{i.})^2}{nMS_E \sum_{i=1}^{a} c_i^2} = \frac{MS_C}{MS_E} = \frac{SS_C/1}{MS_E}$.
Example. In the example with looms, suppose we would like to test the hypotheses $H_0 :\ \mu_2 = (\mu_1 + \mu_3)/2$ and $H_0 :\ \mu_1 = \mu_3$. The sums of squares of the corresponding orthogonal contrasts

$$C_1 = y_{1.} - 2y_{2.} + y_{3.} = 360 - (2)(366) + 380 = 8, \quad \text{and} \quad C_2 = y_{1.} - y_{3.} = -20$$

are

$$SS_{C_1} = \frac{(8)^2}{(4)(6)} = 2.67, \quad \text{and} \quad SS_{C_2} = \frac{(-20)^2}{(4)(2)} = 50.$$

Notice that $SS_{tr} = 52.67 = SS_{C_1} + SS_{C_2}$.
The ANOVA table is

Source	SS	df	MS	F
Looms	52.67	2	26.34	6.41
$C_1 :\ 2\mu_2 = \mu_1 + \mu_3$	(2.67)	1	2.67	0.65
$C_2 :\ \mu_1 = \mu_3$	(50)	1	50	12.17
Error	37	9	4.11	
Total	89.67	11		

Since $F_{0.05,1,9} = 5.12$, we reject $H_0 :\ \mu_1 = \mu_3$ and conclude that the means for loom and one and loom three differ significantly, whereas we accept $H_0 :\ 2\mu_2 = \mu_1 + \mu_3$ and conclude that the mean for loom 2 equals the average of the means for looms 1 and 3.

3.4. Model Adequacy Checking.

In a one-way ANOVA the observations are described by the model $y_{ij} = \mu_i + \varepsilon_{ij}$. The hypotheses testing of no difference in treatment means is valid only if the following assumptions on the model are satisfied: $\varepsilon \overset{iid}{\sim} N(0, \sigma^2)$. There are, in fact, three assumptions: (i) identical Normal distribution of the errors, (ii) independence of the random errors, and (iii) constant variance ("homoscedasticity").

Prior to performing the analysis of variance, it is wise to check the validity of the assumptions ("model diagnostic checking"). It can be done by examining the residuals.

Definition. The residual for ijth observation is $e_{ij} = y_{ij} - \hat{y}_{ij}$ where $\hat{y}_{ij}$ is the fitted value, the estimate of the ijth observation, $\hat{y}_{ij} = \hat{\mu}_i = \bar{y}_{i.}$. Thus, $e_{ij} = y_{ij} - \bar{y}_{i.}$.

3.4.1. The Normality Assumption.

(a) Plot a histogram of the residuals. It should display a bell-shaped figure centered at zero.
(b) Construct a Normal probability plot (QQ-plot) of the residuals. It should display a linear pattern.
(c) Perform normality tests (see SAS output).

3.4.2. Plot of Residuals in Time Sequence.

To verify the independence assumption, plot residuals in time order (if possible). If a pattern is detected, then there is a correlation between the residuals. The pattern might be a run of positive residuals, then a run of negatives; a megaphone a linear pattern. If the plot is structureless, the residuals are independent.

3.4.3. Plot of Residuals Versus Fitted Values.

(a) To verify for equality of variances, plot residuals against the fitted values. Again, the plot should show no structure.
(b) To test the homogeneity of variance, perform the Bartlett test (see pages 84-85 and SAS output). The hypotheses of interest are $H_0: \ \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_a^2$ and $H_1: \ H_0$ is false for at least one σ_i^2.

4.1. The Randomized Complete Block Design.

Example. The quality control engineer suspects that the difference in fabric strength for the three looms is caused by difference in work experience and skills of the three operators. He decides to set up an experiment with operators being the blocking factor, a nuisance source of variability. Within a block, the order in which the three looms are tested is randomly determined. He gets the following results:

	Operators		
Looms	1	2	3
1	88	93	90
2	90	92	92
3	91	96	94

Notice that each loom is tested for each operator, that is, each block contains all treatments. This defines a randomized complete block design.

4-1.1. Statistical Analysis of the RCBD.

Definition. The statistical model for this design is $y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, \quad i = 1, \dots, a, \quad j = 1, \dots, b$, where μ is the overall mean, τ_i is the ith treatment effect, β_j is the jth block effect, and ε_{ij} is a random error. It is assumed that ε_{ij}'s are i.i.d. $N(0, \sigma^2)$. Since τ_i's and β_j's are defined as deviations from the overall mean, the assumptions are imposed such as $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$. Notice that this is an additive model.
We are interested in testing for zero treatment effect, that is, $H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0$ versus $H_1: \tau_i \neq 0$ for some i.
Note. We may also be interested in testing for zero blocking effect since, if $H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$ is accepted, blocking may not be necessary in future experiments.
Derivation of the test statistic.

Notation.

$$y_{i.} = \sum_{j=1}^{b} y_{ij}, \quad \bar{y}_{i.} = \frac{y_{i.}}{b}, \quad i = 1, \dots, a,$$

$$y_{.j} = \sum_{i=1}^{a} y_{ij}, \quad \bar{y}_{.j} = \frac{y_{.j}}{a}, \quad j = 1, \dots, b,$$

$$y_{..} = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij} = \sum_{i=1}^{a} y_{i.} = \sum_{j=1}^{b} y_{.j}, \quad \bar{y}_{..} = \frac{y_{..}}{N}, \ N = ab.$$

We express the total corrected sum of squares as

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^{a} \sum_{j=1}^{b} [(\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) + (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})]^2$$

$$= b \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2 + a \sum_{j=1}^{b} (\bar{y}_{.j} - \bar{y}_{..})^2 + \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{i.})^2$$

$$+2 \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{i.} - \bar{y}_{..})(\bar{y}_{.j} - \bar{y}_{..}) + 2 \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{i.} - \bar{y}_{..})(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{i.})$$

$$+2 \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{.j} - \bar{y}_{..})(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{i.})$$

$= \{$the last three cross products are zero!$\} = SS_{tr} + SS_{bl} + SS_E$
where SS_{tr} is the sum of squares due to treatments, SS_{bl} is the sum of squares due to blocks, and SS_E is the sum of squares due to error. Define the mean squares by $MS_{tr} = \frac{SS_{tr}}{a-1}$, $MS_{bl} = \frac{SS_{bl}}{b-1}$, and $MS_E = \frac{SS_E}{(b-1)(a-1)}$. It can be shown that

$$\mathbb{E}(MS_{tr}) = \sigma^2 + \frac{b \sum_{i=1}^{a} \tau_i^2}{a-1}$$

$$\mathbb{E}(MS_{bl}) = \sigma^2 + \frac{a\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$\mathbb{E}(MS_E) = \sigma^2.$$

To test $H_0 : \tau_i = 0$, we would use the test statistic $F = \frac{MS_{tr}}{MS_E}$ which is distributed as $F_{a-1,(a-1)(b-1)}$ if the null hypothesis is true. Thus, we would reject the H_0, if $F > F_{\alpha,\,a-1,\,(a-1)(b-1)}$.

Remark. It seems that the hypothesis $H_0 : \beta_j = 0$ may be tested by comparing the statistic $F = \frac{MS_{bl}}{MS_E}$ to $F_{\alpha,\,b-1,\,(a-1)(b-1)}$. However, the randomization has been applied only to treatments *within* blocks, making the F distribution questionable. It is advisable, however, to examine the ratio $\frac{MS_{bl}}{MS_E}$. If it is large, the blocking factor has a large effect.

The ANOVA table looks like this

Source of Variation	SS	DF	MS	F
Treatments	$\sum \frac{y_{i.}^2}{b} - \frac{y_{..}^2}{ab}$	$a-1$	$\frac{SS_{tr}}{a-1}$	$\frac{MS_{tr}}{MS_E}$
Blocks	$\sum \frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab}$	$b-1$	$\frac{SS_{bl}}{b-1}$	
Error	$SS_T - SS_{tr} - SS_{bl}$	$(a-1)(b-1)$	$\frac{SS_E}{(a-1)(b-1)}$	
Total	$\sum\sum y_{ij}^2 - \frac{y_{..}^2}{ab}$	$ab-1$		

Example. In our example,

	Operators			
Looms	1	2	3	Totals
1	88	93	90	271
2	90	92	92	274
3	91	96	94	281
Totals	269	281	276	826

$a = b = 3$, $SS_T = \sum\sum y_{ij}^2 - \frac{y_{..}^2}{ab} = 75854 - \frac{682276}{9} \approx 45.56$, $SS_{tr} = \sum \frac{y_{i.}^2}{b} - \frac{y_{..}^2}{ab} = \frac{227478}{3} - \frac{682276}{9} \approx 75826 - 75808.44 = 17.56$, $SS_{bl} = \sum \frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab} \approx \frac{227498}{3} - 75808.44 = 24.23$, $SS_E = SS_T - SS_{tr} - SS_{bl} = 45.56 - 17.56 - 24.23 = 3.77$, $MS_{tr} = \frac{SS_{tr}}{a-1} = \frac{17.56}{2} = 8.78$, $MS_{bl} = \frac{SS_{bl}}{b-1} = \frac{24.23}{2} \approx 12.12$, $MS_E = \frac{SS_E}{(a-1)(b-1)} = \frac{3.77}{4} \approx 0.94$, $F = \frac{MS_{tr}}{MS_E} = \frac{8.78}{0.94} \approx 9.34$. The ANOVA table is

Source of Variation	SS	DF	MS	F
Looms	17.56	2	8.78	9.34
Operators	24.23	2	12.12	
Error	3.77	4	0.94	
Total	45.56	8		

$F_{\alpha,\,a-1,\,(a-1)(b-1)} = F_{0.05,\,2,\,4} = 6.94$. Therefore, we reject $H_0 : \tau_i = 0$ and conclude that the looms produce fabric of different strength. Also, operators seem to differ significantly, since the mean square for blocks is large relative to error.

4.2. The Latin Square Design.

Example. The effect of five different formulations of an explosive mixture on the explosive force is studied. Each formulation is mixed from a batch of raw material that is only large enough for five formulations to be tested. Furthermore, the formulations are prepared by several operators, and there may be substantial difference in the skills and experience of the operators. Thus, there are two blocking factors: batches of raw material and operators. The appropriate design for this problem consists of testing each formulation exactly once in each batch and for each formulation to be prepared exactly once by each of five operators. The resulting design is called a Latin square design. The design is a square arrangement and the five treatments (formulations) are denoted by the Latin letters A, B, C, D, and E (hence, the name "Latin square"). The coded data are

	Operators					$y_{i..}$
Batches	1	2	3	4	5	
1	A=-1	B=-5	C=-6	D=-1	E=-1	-14
2	B=-8	C=-1	D=5	E=2	A=11	9
3	C=-7	D=13	E=1	A=2	B=-4	5
4	D=1	E=6	A=1	B=-2	C=-3	3
5	E=-3	A=5	B=-5	C=4	D=6	7
$y_{..k}$	-18	18	-4	5	9	$10=y_{...}$

Definition. The statistical model for this design is $y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \varepsilon_{ijk}$, $i, j, k = 1, \dots, p$, where y_{ijk} is the observation in the ith row and kth column for the jth treatment, μ is the overall mean, α_i is the ith row effect, τ_j is the jth treatment effect, β_k is the kth column effect, and ε_{ijk} is a random error. It is assumed that ε_{ijk}'s are i.i.d. $N(0, \sigma^2)$. Notice that only two out of the three subscripts are necessary to identify an observation. This model is additive. The assumptions on the effect terms are $\sum_{i=1}^{p} \alpha_i = 0$, $\sum_{j=1}^{p} \tau_j = 0$, and $\sum_{k=1}^{p} \beta_k = 0$.
We are interested in testing for zero treatment effect, that is, $H_0: \tau_1 = \tau_2 = \cdots = \tau_p = 0$ versus $H_1: \tau_j \neq 0$ for some j.
The ANOVA table for the Latin square design is

Source	SS	DF	MS	F
Treatments	$SS_{tr} = \sum_{j=1}^{p} \frac{y_{.j.}^2}{p} - \frac{y_{...}^2}{p^2}$	$p-1$	$\frac{SS_{tr}}{p-1}$	$\frac{MS_{tr}}{MS_E}$
Rows	$SS_{rows} = \sum_{i=1}^{p} \frac{y_{i..}^2}{p} - \frac{y_{...}^2}{p^2}$	$p-1$	$\frac{SS_{rows}}{p-1}$	
Columns	$SS_{col} = \sum_{k=1}^{p} \frac{y_{..k}^2}{p} - \frac{y_{...}^2}{p^2}$	$p-1$	$\frac{SS_{col}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{p^2}$	p^2-1		

If the test statistic $F > F_{\alpha, p-1, (p-2)(p-1)}$, the null is rejected. In addition, we might want to test for no row effect and no column effect by forming the ratios of MS_{row} or MS_{col} to MS_E. However, since the rows and columns represent restrictions on randomization, these tests may not be appropriate.

Example. In our example, $p = 5$, $SS_T = 680 - \frac{10^2}{25} = 676$, $SS_{batches} = \frac{(-14)^2+9^2+5^2+3^2+7^2}{5} - \frac{10^2}{25} = 68$, $SS_{oper} = \frac{(-18)^2+18^2+(-4)^2+5^2+9^2}{5} - \frac{10^2}{25} = 150$.

Formulation	Total
A	$y_{.1.} = 18$
B	$y_{.2.} = -24$
C	$y_{.3.} = -13$
D	$y_{.4.} = 24$
E	$y_{.5.} = 5$

$SS_{form} = \frac{18^2+(-24)^2+13^2+24^2+5^2}{5} - \frac{10^2}{25} = 330$, $SS_E = 128$. The ANOVA table is

Source	SS	DF	MS	F
Formulations	330	4	82.5	7.73
Batches	68	4	17	
Operators	150	4	37.5	
Error	128	12	10.67	
Total	676	24		

$F = 7.73 > F_{.05,4,12} = 3.26$, therefore, we reject H_0 and conclude that there is a significant difference between the formulations. Also, $\frac{MS_{oper}}{MS_E} = 3.51$, indicating that there are differences between operators, while $\frac{MS_{batches}}{MS_E} = 1.59$ implies no difference between batches. So, blocking on operators is reasonable, whereas blocking on batches is unnecessary (though advisable).

4.3. The Græco-Latin Square Design.

The Græco-Latin square design is used to block in three directions. It can be viewed as superposition of two Latin square designs in which treatments are denoted by Latin letters, and the rows, columns, and Greek letters represent blocks. Each Greek letter appears exactly once with each Latin letter.

Example. Suppose that in the explosive force example, test assemblies is an additional blocking factor, so we run a Græco-Latin square design. The design is

	Operators					$y_{i...}$
Batches	1	2	3	4	5	
1	Aα=-1	Bγ=-5	Cε=-6	Dβ=-1	Eδ=-1	-14
2	Bβ=-8	Cδ=-1	Dα=5	Eγ=2	Aε=11	9
3	Cγ=-7	Dε=13	Eβ=1	Aδ=2	Bα=-4	5
4	Dδ=1	Eα=6	Aγ=1	Bε=-2	Cβ=-3	3
5	Eε=-3	Aβ=5	Bδ=-5	Cα=4	Dγ=6	7
$y_{...k}$	-18	18	-4	5	9	10=$y_{....}$

Definition. The statistical model for this design is $y_{ijkl} = \mu + \theta_i + \tau_j + \omega_k + \Psi_l + \varepsilon_{ijkl}$, $i,j,k,l = 1,\dots,p$, where y_{ijkl} is the observation in the

ith row, Latin letter j, Greek letter k, and lth column, μ is the overall mean, θ_i is the ith row effect, τ_j is the effect of Latin letter j treatment, ω_k is the effect of Greek letter k blocking factor, Ψ_l is the lth column effect, and ε_{ijkl}'s are i.i.d. $N(0, \sigma^2)$ errors. Two subscripts suffice to identify an observations. Græco-Latin square designs exist for all $p \geq 3$ except $p = 6$. The model is additive. The assumptions on the effect terms are $\sum_{i=1}^{p} \theta_i = 0$, $\sum_{j=1}^{p} \tau_j = 0$, $\sum_{k=1}^{p} \omega_k = 0$, and $\sum_{l=1}^{p} \Psi_l = 0$.
We are interested in testing for zero treatment effect, that is, $H_0: \tau_1 = \tau_2 = \cdots = \tau_p = 0$ versus $H_1: \tau_j \neq 0$ for some j.
The ANOVA table for the Græco-Latin square design is

Source	SS	DF	MS	F
Treatments	$SS_L = \sum_{j=1}^{p} \frac{y_{.j..}^2}{p} - \frac{y_{....}^2}{p^2}$	$p-1$		$\frac{MS_L}{MS_E}$
Greek letter	$SS_G = \sum_{k=1}^{p} \frac{y_{..k.}^2}{p} - \frac{y_{....}^2}{p^2}$	$p-1$		
Rows	$SS_{rows} = \sum_{i=1}^{p} \frac{y_{i...}^2}{p} - \frac{y_{....}^2}{p^2}$	$p-1$		
Columns	$SS_{col} = \sum_{l=1}^{p} \frac{y_{...l}^2}{p} - \frac{y_{....}^2}{p^2}$	$p-1$		
Error	SS_E (by subtraction)	$(p-3)(p-1)$		
Total	$SS_T = \sum_i \sum_j \sum_k \sum_l y_{ijkl}^2 - \frac{y_{....}^2}{p^2}$	$p^2 - 1$		

If the test statistic $F > F_{\alpha, p-1, (p-3)(p-1)}$, the null is rejected. In addition, we might want to test for no row effect, or no column effect, or no Greek letter factor effect by forming the ratios of MS_{rows} or MS_{col} or MS_G to MS_E. However, since the blocks represent restrictions on randomization, these tests may not be appropriate.

Example. In our example, $p = 5$, $SS_T = 676$, $SS_{batches} = 68$, $SS_{oper} = 150$, $SS_{form} = 330$.
The total for the test assemblies are

Greek letter	Total
α	10
β	-6
γ	-3
δ	-4
ε	13

The sum of squares due to assemblies is $SS_{assem} = 330/5 - 100/25 = 62$. The error sum of squares is $SS_E = 66$. The ANOVA table is

Source	SS	DF	MS	F
Formulations	330	4	82.5	10.0
Batches	68	4	17	
Operators	150	4	37.5	
Assemblies	62	4	15.5	
Error	66	8	8.25	
Total	676	24		

$F = 10 > F_{.05,4,8} = 3.84$, therefore, we reject H_0 and conclude that there is a significant difference between the formulations. Also, $\frac{MS_{oper}}{MS_E} = 4.55$, indicating that there are differences between operators, while $\frac{MS_{batches}}{MS_E} = 2.06$ and $\frac{MS_{assem}}{MS_E} = 1.88$ imply no difference between batches or test assemblies. So, blocking on operators is reasonable, whereas blocking on batches and assemblies is unnecessary (though advisable).

4.4. The Balanced Incomplete Block Design.

Example. A chemical engineer suspects that the time of reaction for a chemical process is a function of the type of catalyst employed. Four catalysts are investigated. The experiment consists of taking a batch of raw material, loading the pilot plant, applying each catalyst separately, and observing the reaction time. The batches of the raw material are used as blocks. The order in which the catalysts are run in each block is randomized. However, each batch is only large enough for three catalysts to be run. Therefore, an incomplete block design must be used.
Definition. An incomplete block design is a randomized block design, in which every treatment is not present in every block.
Definition. A balanced incomplete block design is an incomplete block design in which any two treatments appear together an equal number of times.
Example (continued...). The balanced incomplete block design for this experiment is

	Batch of Raw Material				
Catalyst	1	2	3	4	Totals
1	73	74	–	71	218
2	–	75	67	72	214
3	73	75	68	–	216
4	75	–	72	75	222
Totals	221	224	207	218	870

4-4.1. Statistical Analysis of the BIBD.

Notation. We assume that there are a treatments and b blocks. In addition, we assume that each block contains k treatments, that each treatment occurs r times in the design, and that there are $N = ar = bk$ total observations.
Proposition. The number of times each pair of treatments appears in the same block is $\lambda = \frac{r(k-1)}{a-1}$.
Proof: Consider, for example, treatment 1. It appears in r blocks and there are $k-1$ other treatments in each of those blocks. Therefore, there are $r(k-1)$ additional observations in the design, which must represent the remaining $a-1$ treatments λ times. Thus, $(a-1)\lambda = r(k-1)$. □
Definition. The statistical model for this design is $y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$, where μ is the overall mean, τ_i is the ith treatment effect, β_j is the jth block effect, and ε_{ij} is a random error. It is assumed that ε_{ij}'s are i.i.d.

$N(0, \sigma^2)$. This is an additive model. The assumptions on the effect terms are $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$.
We are interested in testing for zero treatment effect, that is, $H_0 : \tau_1 = \tau_2 = \cdots = \tau_a = 0$ versus $H_1 : \tau_i \neq 0$ for some i.
Derivation of the test statistic.

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{N} = SS_{tr(adj)} + SS_{bl} + SS_E$$

where the sum of squares for treatments is adjusted to separate the treatment and the block effects. This adjustment is necessary because each treatment is represented in a different set of r blocks.
The block sum of squares $SS_{bl} = \sum_{j=1}^{b} \frac{y_{.j}^2}{k} - \frac{y_{..}^2}{N}$ has $b-1$ degrees of freedom. The adjusted treatment sum of squares is

$$SS_{tr(adj)} = \frac{k \sum_{i=1}^{a} Q_i^2}{\lambda\, a}$$

where $Q_i = y_{i.} - \frac{1}{k} \sum_{j=1}^{b} n_{ij} y_{.j},\ i = 1, \ldots, a$ with $n_{ij} = 1$ if treatment i appears in block j and $n_{ij} = 0$ otherwise. Q_i's are called the adjusted treatment totals and always sum up to zero.
$SS_{tr(adj)}$ has $a-1$ degrees of freedom. The error sum of squares is computed by subtraction; it has $N - a - b + 1$ degree of freedom. The test statistic is $F = \frac{MS_{tr(adj)}}{MS_E}$. The null hypothesis is rejected if $F > F_{\alpha, a-1, N-a-b+1}$.

The ANOVA table looks like this

Source of Variation	SS	DF	MS	F
Treatments (adj)	$\frac{k \sum Q_i^2}{\lambda a}$	$a-1$	$\frac{SS_{tr(adj)}}{a-1}$	$\frac{MS_{tr(adj)}}{MS_E}$
Blocks	$\sum \frac{y_{.j}^2}{k} - \frac{y_{..}^2}{N}$	$b-1$		
Error	$SS_T - SS_{tr(adj)} - SS_{bl}$	$N-a-b+1$	$\frac{SS_E}{N-a-b+1}$	
Total	$\sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$	$N-1$		

Remark. We might be interested in testing $H_0 : \beta_j = 0\, \forall j$. To do so, compute $SS_{tr} = \sum_{i=1}^{a} \frac{y_{i.}^2}{r} - \frac{y_{..}^2}{N}$ and $SS_{bl(adj)} = SS_T - SS_{tr} - SS_E$. If the ratio $\frac{SS_{bl(adj)}/(b-1)}{SS_E/(N-a-b+1)}$ is large, we reject the null.
Example. In our example, $a = 4,\ b = 4,\ r = 3,\ k = 3,\ \lambda = \frac{3(3-1)}{4-1} = 2,\ N = (4)(3) = 12$. The sum of squares are $SS_T = \sum_i \sum_j y_{ij}^2 - \frac{y_{..}^2}{12} = 63,156 - \frac{(870)^2}{12} = 81,\ SS_{bl} = \sum_j \frac{y_{.j}^2}{3} - \frac{y_{..}^2}{12} = \frac{(221)^2+(207)^2+(224)^2+(218)^2}{3} - \frac{(870)^2}{12} = 55$. The adjusted treatment totals are $Q_1 = 218 - \frac{1}{3}(221 + 224 + 218) = -9/3,\ Q_2 = 214 - \frac{1}{3}(224+207+218) = -7/3,\ Q_3 = 216 - \frac{1}{3}(221+224+207) = -4/3,\ Q_4 = 222 - \frac{1}{3}(221+207+218) = 20/3$. The adjusted sum of squares for treatments is $SS_{tr(adj)} = \frac{k \sum_i Q_i^2}{\lambda\, a} = \frac{(3)((-9/3)^2+(-7/3)^2+(-4/3)^2+(20/3)^2)}{(2)(4)} = 22.75$. The error sum of squares is $SS_E = SS_T - SS_{bl} - SS_{tr(adj)} = 81 - 55 - 22.75 = 3.25$. The ANOVA table is

Source of Variation	SS	DF	MS	F
Catalysts	22.75	3	7.58	11.66
Batches	55.00	3		
Error	3.25	5	0.65	
Total	81.00	11		

The test statistic $F = 11.66 > F_{0.05,3,5} = 5.41$, therefore, the null hypothesis of no treatment effect is rejected, and we conclude that the catalyst employed has a significant effect on the time of reaction. Now, to test whether there is a blocking effect, compute

$$SS_{tr} = 11.67, \; SS_E = 3.25, \; SS_{bl(adj)} = 66.08$$

$$\Longrightarrow \quad \frac{SS_{bl(adj)}/(b-1)}{SS_E/(N-a-b+1)} = 33.89.$$

We conclude that it is reasonable to use batches as the blocking factor.

5.3. The Two-Factor Factorial Design.

Suppose there are two treatments (factors) A and B. An experiment is run for all possible combinations of the levels of the factors, and there are, in general, more than one replicate in each cell.

Example. It is suspected that temperature and concentration might affect the yield of a chemical process. The data are

	Concentration		
Temperature	40	60	80
50	17	16	24
	20	21	22
75	12	18	17
	9	13	12
100	16	18	25
	12	21	23

5.3.2. Statistical Analysis of the Fixed Effects Model.

Definition. The statistical model is $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}, \; i = 1, \dots, a, \; j = 1, \dots, b, \; k = 1, \dots, n$, where μ is the overall mean, τ_i is the effect of the ith level of treatment A, β_j is the effect of the jth level of treatment B (main effects), $(\tau\beta)_{ij}$ is the effect of the interaction between τ_i and β_j, and $\varepsilon_{ijk} \sim N(0, \sigma^2)$ is a random error. Notice that this model is not additive.

The assumptions on the effect terms are $\sum_{i=1}^{a} \tau_i = 0, \; \sum_{j=1}^{b} \beta_j = 0,$

$\sum_{i=1}^{a}(\tau\beta)_{ij} = 0,\ j = 1, \ldots, b$, and $\sum_{j=1}^{b}(\tau\beta)_{ij} = 0,\ i = 1, \ldots, a$.
The hypotheses of interest are $H_0 : \tau_i = 0\ \forall i$, $H_0 : \beta_j = 0\ \forall j$, or $H_0 : (\tau\beta)_{ij} = 0\ \forall i, j$. The ANOVA table for the analysis is

Source	SS	DF	F
A	$\sum_{i=1}^{a} \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}$	$a-1$	$\frac{MS_A}{MS_E}$
B	$\sum_{j=1}^{b} \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}$	$b-1$	$\frac{MS_B}{MS_E}$
AB	$\sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\frac{MS_{AB}}{MS_E}$
Error	$SS_T - SS_A - SS_B - SS_{AB}$	$ab(n-1)$	
Total	$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$	

Example (continued...). The ANOVA table is

Source	SS	DF	MS	F
Temperature	150.11	2	75.06	10.72
Concentration	114.78	2	57.39	8.20
Interaction	40.55	4	10.14	1.45
Error	63	9	7	
Total	368.44	17		

$F_{0.05,2,9} = 4.26$, $F_{0.05,4,9} = 3.63$, therefore, we conclude that there are main effects, but no interaction present.

The Main Effect Plots and the Interaction Plot.

Proposition. The estimated effect of the ith level of factor A is $\bar{y}_{i..}$, that of the jth level of factor B is $\bar{y}_{.j.}$, and that of their interaction is $\bar{y}_{ij.}$.
Proof: The model parameter estimates for a two-factor factorial design are (see Section 5-3.4) $\hat{\mu} = \bar{y}_{...}$, $\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$, $\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$, and $\widehat{(\tau\beta)}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$.
The estimated main effect of the ith level of factor A is $\hat{\mu} + \hat{\tau}_i = \bar{y}_{...} + \bar{y}_{i..} - \bar{y}_{...} = \bar{y}_{i..}$. Likewise, the effect of the jth level of factor B is estimated by $\bar{y}_{.j.}$. The estimate of the observation in the ijth cell is given by $\hat{y}_{ijk} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + \widehat{(\tau\beta)}_{ij} = \bar{y}_{ij.}$.

Definition. A main effect plot displays the estimated values for different levels of a factor.
In our example, the estimates for factor A are $\bar{y}_{1..} = 20$, $\bar{y}_{2..} = 13.5$, $\bar{y}_{3..} = 19.17$, and the estimates for factor B are $\bar{y}_{.1.} = 14.33$, $\bar{y}_{.2.} = 17.83$, $\bar{y}_{.3.} = 20.5$
Definition. An interaction plot displays the estimated values for different treatment-treatment combinations.
In our example, these estimated values are $\bar{y}_{11.} = 18.5$, $\bar{y}_{12.} = 18.5$, $\bar{y}_{13.} = 23.0$, $\bar{y}_{21.} = 10.5$, $\bar{y}_{22.} = 15.5$, $\bar{y}_{23.} = 14.5$, $\bar{y}_{31.} = 14.0$, $\bar{y}_{32.} = 19.5$, $\bar{y}_{33.} = 24.0$.

6.2. The 2^2 Factorial Design.

Definition. The 2^2 factorial design is a design with two factors A and B, each at two levels. The levels of the factors may be arbitrarily called "low" and "high."

Example. In a chemical reaction, the reactant concentration is factor A run at two levels, 15% and 25%, and the catalyst is factor B, with two levels, one bag used or two bags used. The experiment is replicated three times, and the data are

	Replicate			
Treatments	I	II	III	Total
A low, B low	28	25	27	$(1) = 80$
A high, B low	36	32	32	$a = 100$
A low, B high	18	19	23	$b = 60$
A high, B high	31	30	29	$ab = 90$

The effect of treatment A at the low level of B is $[a-(1)]/n$ and the effect of A at the high level of B is $[ab-b]/n$. Therefore, the main effect of treatment A is the average of this two quantities

$$A = \frac{1}{2n}[ab-b]+[a-(1)] = \frac{1}{2n}[a+ab-(1)-b] = \frac{[\text{contrast}]}{2n}.$$

Likewise, the main effect of treatment B is

$$B = \frac{1}{2n}[ab-a]+[b-(1)] = \frac{1}{2n}[b+ab-(1)-a].$$

The interaction between A and B is the average difference between the effect of A at the high level of B and the effect of A at the low level of B. Thus,

$$AB = \frac{1}{2n}[ab-b]-[a-(1)] = \frac{1}{2n}[(1)+ab-a-b].$$

Remark. Alternatively, AB can be defined as the average difference between the effect of B at the high level of A and the effect of B at the low level of A. Both definitions lead to the same equation.

Example (continued...) $A = \frac{1}{2(3)}(100+90-80-60) = 8.33$, $B = \frac{1}{2(3)}(60+90-80-100) = -5.00$, $AB = \frac{1}{2(3)}(80+90-100-60) = 1.67$.

To test for main and interaction effects, compute the sum of squares

$$SS = \frac{[\text{contrast}]^2}{4n}, \quad SS_T = \sum\sum\sum y_{ijk}^2 - \frac{y_{...}^2}{4n},$$
$$SS_E = SS_T - SS_A - SS_B - SS_{AB}.$$

Example (continued...) The ANOVA table is

Source	SS	DF	MS	F
A	208.33	1	208.33	53.15
B	75.00	1	75.00	19.13
AB	8.33	1	8.33	2.13
Error	31.34	8	3.92	
Total	323.00	11		

$F_{0.05,1,8} = 5.32$. There are main effects of A and B but no interaction.
Remark. To coefficients for estimating the effects are

Effect	(1)	a	b	ab
A	-1	+1	-1	+1
B	-1	-1	+1	+1
AB	+1	-1	-1	+1

The coefficients for estimating AB is the product of the corresponding coefficients for the two main effects.
Example. The following data were obtained by running four replicates of a 2^2 design. Analyze the data and draw conclusions.

		Replicate			
A	B	I	II	III	IV
-	-	18.2	18.9	12.9	14.4
+	-	27.2	24.0	22.4	22.5
-	+	15.9	14.5	15.1	14.2
+	+	41.0	43.0	36.3	39.9

6.3. The 2^3 Factorial Design.

Definition. The 2^3 factorial design is a design with three factors, A, B, and C, each at two levels.
Algebraic signs for calculating the effects in the model are:

Treatment	Factorial Effect							
Combination	I	A	B	AB	C	AC	BC	ABC
(1)	+	-	-	+	-	+	+	-
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
ab	+	+	+	+	-	-	-	-
c	+	-	-	+	+	-	-	+
ac	+	+	-	-	+	+	-	-
bc	+	-	+	-	+	-	+	-
abc	+	+	+	+	+	+	+	+

For example,

$$A = \frac{1}{4n}[a + ab + ac + abc - (1) - b - c - bc] = \frac{[\text{contrast}]}{4n}.$$

To test the main and interaction effects, compute

$$SS = \frac{[\text{contrast}]^2}{8n}, \quad SS_T = \sum\sum\sum\sum y_{ijkl}^2 - \frac{y_{....}^2}{8n}, \quad SS_E \text{ by subtraction.}$$

Example. The following data were obtained by running two replicates of a 2^3 design. Analyze the data and draw conclusions.

Treatment Combination	Replicate I	II
(1)	18.2	18.9
a	12.9	14.4
b	27.2	24.0
ab	22.4	22.5
c	15.9	14.5
ac	15.1	14.2
bc	41.0	43.0
abc	36.3	39.9

5.6. Blocking in a Factorial Design.

Example. It is suspected that temperature and concentration might affect the yield of a chemical process. There is enough time to run one replicate of the experiment in one day, so days represent blocks. The order in which the experiment is run within each block is randomized. The data are

	Day 1 Concentration			Day 2 Concentration		
Temperature	40	60	80	40	60	80
50	17	16	24	20	21	22
75	12	18	17	9	13	12
100	16	18	25	12	21	23

Statistical Analysis of a Two-Factor Factorial Design with Blocking. Suppose n replicates of a two-factor factorial design are run in n blocks each containing a single replicate. The model for this experiment is $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \delta_k + \varepsilon_{ijk},\ i = 1, \dots, a,\ j = 1, \dots, b,\ k = 1, \dots, n$ where δ_k is the effect of the kth block. Notice that the interaction between the block and the treatments is assumed negligibly small, that is, the terms $(\tau\delta)_{ik}$, $(\beta\delta)_{jk}$, and $(\tau\beta\delta)_{ijk}$ are assumed equal to zero. The assumptions on the effect terms are $\sum_{i=1}^a \tau_i = 0$, $\sum_{j=1}^b \beta_j = 0$, $\sum_{i=1}^a (\tau\beta)_{ij} = 0$, $\sum_{j=1}^b (\tau\beta)_{ij} = 0$, and $\sum_{k=1}^n \delta_k = 0$.
The ANOVA table is

Source	SS	DF	F
A	$\sum_{i=1}^a \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}$	$a-1$	$\frac{MS_A}{MS_E}$
B	$\sum_{j=1}^b \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}$	$b-1$	$\frac{MS_B}{MS_E}$
AB	$\sum_{i=1}^a \sum_{j=1}^b \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\frac{MS_{AB}}{MS_E}$
Blocks	$\sum_{k=1}^n \frac{y_{..k}^2}{ab} - \frac{y_{...}^2}{abn}$	$n-1$	$\left(\frac{MS_{blocks}}{MS_E}\right)$
Error	subtraction	$(ab-1)(n-1)$	
Total	$\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$	

Example (continued...). $y_{..1} = 163$, $y_{..2} = 153$, $SS_{blocks} = (163)^2/9 + (153)^2/9 - (316)^2/18 = 5.56$.The ANOVA table is

Source	SS	DF	MS	F
Temperature	150.11	2	75.06	10.45
Concentration	114.78	2	57.39	7.99
Interaction	40.55	4	10.14	1.41
Days	5.56	1	5.56	(0.77)
Error	57.44	8	7.18	
Total	368.44	17		

$F_{0.05,2,8} = 4.46$, $F_{0.05,4,8} = 3.84$, therefore, we conclude that there are main effects, but no interaction present.

7.2. Blocking a Replicated 2^k Design.

Example. In a chemical reaction, the reactant concentration is factor A run at two levels, and the catalyst is factor B, with two levels. Only four experiments can be made from a single batch of raw material. Therefore, three batches (blocks) are used to run three replicates. The data are

	Batch		
Treatments	1	2	3
(1)	28	25	27
a	36	32	32
b	18	19	23
ab	31	30	29
Total	113	106	111

The sum of squares for the main effects and the interactions are computed as in the 2^k factorial design. The sum of squares for blocks is $SS_{blocks} = \sum_{i=1}^{n} \frac{B_i^2}{2^k} - \frac{y_{...}^2}{n2^k}$. The sum of squares for error is calculated by subtraction. In our example, $SS_{blocks} = 6.5$. The ANOVA table is

Source	SS	DF	MS	F
A	208.33	1	208.33	50.32
B	75.00	1	75.00	18.12
AB	8.33	1	8.33	2.01
Blocks	6.5	2	6.5	(1.57)
Error	24.84	6	4.14	
Total	323.00	11		

$F_{0.05,\,1,\,6} = 5.99$. There are main effects of A and B but no interaction.

7.3. Confounding in the 2^k Factorial Design.

Definition. Sometimes a complete replicate of a 2^k factorial design cannot be run within one block. Then the design has to be run in 2, 4, 8, or, in general, 2^p incomplete blocks ($p < k$). In this case certain treatment effects (usually high-order interactions) cannot be distinguished from blocks or are confounded with blocks.

7.4. Confounding in the 2^k Factorial Design in Two Blocks.

Example. In a chemical reaction, the reactant concentration is factor A run at two levels, and the catalyst is factor B, with two levels. Three replicates of the 2^2 design are to be run. Only two experiments can be made from a single batch of raw material. Therefore, six batches (incomplete blocks) are used to run the replicates. The data are

Replicate I		Replicate II		Replicate III	
Batch 1	Batch 2	Batch 3	Batch 4	Batch 5	Batch 6
(Block 1)	(Block 2)	(Block 1)	(Block 2)	(Block 1)	(Block 2)
$(1) = 28$	$a = 36$	$(1) = 25$	$a = 32$	$(1) = 27$	$a = 32$
$ab = 31$	$b = 18$	$ab = 30$	$b = 19$	$ab = 29$	$b = 23$

The sum of squares for blocks is computed by $SS_{blocks} = \sum \frac{B_i^2}{2} - \frac{\bar{y}_{\cdots}^2}{12}$ where B_i is the total for the ith block. Consider the main effects $A = \frac{1}{2n}[a+ab-(1)-b]$ and $B = \frac{1}{2n}[b+ab-(1)-a]$. The treatment combinations a and b, and (1) and ab have opposite signs. Since they come from different blocks, the main effects are unaffected by the blocking. The interaction AB, however, is confounded with the blocks since in $AB = \frac{1}{2n}[(1) + ab - a - b]$, a and b, and (1) and ab have the same signs. This means that $SS_T = SS_{blocks} + SS_A + SS_B + SS_{error}$. The ANOVA table is

Source	SS	DF	MS	F
A	208.33	1	208.33	36.76
B	75.00	1	75.00	13.24
Blocks	17.00	5	3.4	(0.60)
AB	8.33	1		
replicates		2		
Error for ABC (replicates$\times$blocks)		2		
Error	22.67	4	5.67	
replicates$\times$effects				
Total	323.00	11		

How to assign treatment combinations to the two blocks in the 2^k design with a certain combination being confounded?

There are a number of methods to assign treatments to blocks. One method is to look at the table of plus and minus signs for the 2^k design and assign the treatment combinations that are plus on the confounded combination to block 1, for example, and those that are minus on that combination to block 2.

Example. An experimenter wants to run four replicates of 2^3 design in two blocks with ABC confounded. He should assign treatment combinations a, b, c, and abc to block 1, and $(1), ab, ac$, and bc to block 2.

Another method is to consider a defining contrast $L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$ where $\alpha_i = 0$ or 1 depending on whether the ith treatment is present in the confounded treatment combination. In our example, the defining contrast corresponding to ABC is $L = x_1 + x_2 + x_3$.

The rule for placing treatment combinations into the two blocks is the following. The treatment combinations that produce the same value of $L(mod\ 2)$ are placed in the same block. In our example,

$(1) \to x_1 = 0,\ x_2 = 0,\ x_3 = 0 \to L = 0 \to$ Block 1
$a \to x_1 = 1,\ x_2 = 0,\ x_3 = 0 \to L = 1 \to$ Block 2
$b \to x_1 = 0,\ x_2 = 1,\ x_3 = 0 \to L = 1 \to$ Block 2
$c \to x_1 = 0,\ x_2 = 0,\ x_3 = 1 \to L = 1 \to$ Block 2
$ab \to x_1 = 1,\ x_2 = 1,\ x_3 = 0 \to L = 0 \to$ Block 1
$ac \to x_1 = 1,\ x_2 = 0,\ x_3 = 1 \to L = 0 \to$ Block 1
$bc \to x_1 = 0,\ x_2 = 1,\ x_3 = 1 \to L = 0 \to$ Block 1
$abc \to x_1 = 1,\ x_2 = 1,\ x_3 = 1 \to L = 1 \to$ Block 2

Example. Set up a 2^4 design in two blocks with $ABCD$ confounded.
Solution: The defining contrast $L = x_1 + x_2 + x_3 + x_4$. Therefore, treatment combinations $(1), ab, ac, bc, ad, bd, cd$, and $abcd$ are assigned to block 1, and the others to block 2.

7.6. Confounding in the 2^k Factorial Design in Four Blocks.
The 2^k factorial designs ($k \geq 3$) can be run in four blocks with 2^{k-2} observations each. In this case one has to specify two effects to be confounded with blocks.

Example. Construct a 2^4 factorial design in four blocks with ABC and BCD confounded.
Solution: The defining contrasts are $L_1 = x_1 + x_2 + x_3$ and $L_2 = x_2 + x_3 + x_4$.
$L_1 = 0,\ L_2 = 0$ for $(1), bc, abd, acd$
$L_1 = 1,\ L_2 = 0$ for $b, c, ad, abcd$
$L_1 = 0,\ L_2 = 1$ for d, ab, ac, bcd
$L_1 = 1,\ L_2 = 1$ for a, bd, cd, abc

Remark. Another effect must be confounded. Since four blocks have three

degrees of freedom, three effects should be confounded with blocks. To find the third confounded effect compute the generalized interaction which is defined as the product of the two effects modulus 2. Since the confounded effects are constant in a block (equal either 1 or -1), their product is a constant in a block (either 1 or -1), and, thus, is confounded with the block.
Example. Also $(ABC)(BCD) = AD$ must be confounded.

7.7. Confounding in the 2^k Factorial Design in 2^p Blocks.

In general, the 2^k factorial design can be run in 2^p $(p < k)$ blocks with 2^{k-p} observations each. Then p effects are confounded with the blocks. The assignment of the treatments to the blocks can be made by writing the p defining contrasts $L_1, L_2, \ldots, L_p$ associated with the confounded effects. In addition, $2^p - p - 1$ other effects will be confounded with blocks. These are the generalized interactions of the p initially specified effects.

7.8. Partial Confounding.

A confounded effect cannot be estimated, therefore, a partially confounded design is used. In a partially confounded design, different effects are confounded for different replicates of the design. This way all the effects can be estimated.
Example. For the data considered early, confound A in replicate I, B in replicate II, and AB in replicate III. The data are

Replicate I		Replicate II		Replicate III	
Batch 1 (Block 1)	Batch 2 (Block 2)	Batch 3 (Block 1)	Batch 4 (Block 2)	Batch 5 (Block 1)	Batch 6 (Block 2)
$(1) = 28$	$a = 36$	$(1) = 25$	$b = 19$	$(1) = 27$	$a = 32$
$b = 18$	$ab = 31$	$a = 32$	$ab = 30$	$ab = 29$	$b = 23$

The ANOVA table is

Source	SS	DF	
A (from replicates II and III)	105.125	1	
B (from replicates I and III)	60.5	1	
AB (from replicates I and II)	10.125	1	
Blocks	17.00	5	
Replicates		2	...
A (rep.I)+B(rep.II)+AB(rep.III)		3	
Error	130.25	3	
Total	323.00	11	

8.2. The One-Half Fraction of the 2^k Design.

Sometimes it is very expensive to run a whole replicate of the 2^k design.

In this case the fractional factorial design can be run, which is a fraction of the complete design.

Example. Suppose that experimenters cannot afford all eight runs of a 2^3 design, but can afford four runs. Therefore, a one-half fraction of the 2^3 design is run. It is called 2^{3-1} design.

Suppose treatments a, b, c, and abc are chosen for the experiment. From the table of plus and minus signs, these treatments correspond to the plus signs in the ABC column. The combination ABC is called the generator of this fraction. Because the identity I always has the positive sign, the defining relation for this fraction is $I = ABC$.

Treatment	Factorial Effect							
Combination	I	A	B	AB	C	AC	BC	ABC
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
c	+	-	-	+	+	-	-	+
abc	+	+	+	+	+	+	+	+
(1)	+	-	-	+	-	+	+	-
ab	+	+	+	+	-	-	-	-
ac	+	+	-	-	+	+	-	-
bc	+	-	+	-	+	-	+	-

From the table, the linear combinations used to estimate the main effects of A, B, and C are

$$l_A = \frac{a - b - c + abc}{2} = l_{BC}, \; l_B = \frac{-a + b - c + abc}{2} = l_{AC},$$

and

$$l_C = \frac{-a - b + c + abc}{2} = l_{AB}.$$

Thus, the effects of A and BC, B and AC, and C and AB are indistinguishable. These pairs are called aliases. When estimating A, we are really estimating $A + BC$, etc.

To find the alias to a treatment combination, multiply (modulus 2) the defining relation by this treatment. For example, $A \cdot I = A \cdot ABC = BC$.

A fractional factorial designs are set up in such a way that by running additional fractional designs, one gets a complete replicate of a factorial design.

Example. Terms $(1), ab, ac$, and bc could have been chosen for the one-half fractional design. The defining relation of this design is $I = -ABC$. The aliases are $AB \cdot I = -AB \cdot ABC = -C$, $AC \cdot I = -B$, and $BC \cdot I = -A$. Thus, by estimating A, for example, we really estimate $A - BC$.

Notice that combining the two described one-half fractions, we get a complete replicate of the 2^3 design.

Example. A 2^{3-1} design with defining relation $I = ABC$ produced the following data. Find the important effects.

Treatment	Observation
a	3
b	4
c	7
abc	5

$$SS_{A+BC} = \frac{(a-b-c+abc)^2}{4} = 9/4 = 2.25, SS_{B+AC} = 1/4 = 0.25,$$
$$SS_{C+AB} = 25/4 = 6.25.$$

The main effects of C and possibly A are important.

Example. Set up a 2^{4-1} design with the defining relation $I = -ABCD$. Find the aliases for the main effects.
Solution: The treatments used for the design are $a, b, c, d, abc, abd, acd$, and bcd since they have minus signs in the $ABCD$ column.

13.1. The Random Effects One-Way ANOVA Model.

Definition. A random effects one-way ANOVA model is the model in which the a treatments are a random sample from a larger population of treatments. The population size is assumed infinite.
The statistical model is $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, where ε_{ij}'s are i.i.d. $N(0, \sigma^2)$, τ_i's are i.i.d. $N(0, \sigma_\tau^2)$, and τ_i and ε_{ij} are independent.
The variances σ^2 and σ_τ^2 are called the variance components. We are interested in testing for zero treatment effect, that is, we would like to test $H_0 : \sigma_\tau^2 = 0$ against $H_1 : \sigma_\tau^2 > 0$.
It can be shown that $\mathbb{E}(MS_{tr}) = \sigma^2 + n\sigma_\tau^2$, and $\mathbb{E}(MS_E) = \sigma^2$. Thus, under the null, MS_{tr} and MS_E estimate the same thing, and the test statistic $F = MS_{tr}/MS_E$ has the F distribution with $a-1$ and $N-a$ degrees of freedom.
The estimators of the variance components are $\hat{\sigma}^2 = MS_E$ and $\hat{\sigma}_\tau^2 = (MS_{tr} - MS_E)/n$.
Example. A manufacturer suspects that the batches of raw material furnished by his supplier differ significantly in calcium content. There is a large number of batches currently in the warehouse. Five of these are randomly selected for study. Three observations are taken on each batch. The computed sums of squares are $SS_{tr} = 5.16$ and $SS_E = 1.13$.
(a) Is there significant variation in calcium content from batch to batch?
$F = 11.42,\ F_{0.05,4,10} = 3.48$
(b) Estimate the variance components.
$\hat{\sigma}^2 = 0.113,\ \hat{\sigma}_\tau^2 = 0.39$

13.2. The Two-Factor Factorial with Random Effects.

In this model, a levels of factor A and b levels of factor B are randomly chosen. The model is $y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ij}$, where τ_i, β_j, $(\tau\beta)_{ij}$, and ε_{ij} are independent normal random variables with mean zero and variances, respectively, σ^2_τ, σ^2_β, $\sigma^2_{\tau\beta}$, and σ^2. We are interested in testing $H_0 : \sigma^2_\tau = 0$ against $H_1 : \sigma^2_\tau > 0$; $H_0 : \sigma^2_\beta = 0$ against $H_1 : \sigma^2_\beta > 0$; and $H_0 : \sigma^2_{\tau\beta} = 0$ against $H_1 : \sigma^2_{\tau\beta} > 0$.
It can be shown that $\mathbb{E}(MS_A) = \sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_\tau$, $\mathbb{E}(MS_B) = \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_\beta$, $\mathbb{E}(MS_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}$, and $\mathbb{E}(MS_E) = \sigma^2$.
Thus, the test statistic for $H_0 : \sigma^2_{\tau\beta} = 0$ is $F = MS_{AB}/MS_E$ which has the F distribution with $(a-1)(b-1)$ and $ab(n-1)$ degrees of freedom.
The statistic for $H_0 : \sigma^2_\tau = 0$ is $F = MS_A/MS_{AB}$ which has the F distribution with $a-1$ and $(a-1)(b-1)$ degrees of freedom.
For testing $H_0 : \sigma^2_\beta = 0$ the test statistic is $F = MS_B/MS_{AB}$ which has the F distribution with $b-1$ and $(a-1)(b-1)$ degrees of freedom.
The estimators of the variance components are $\hat{\sigma}^2 = MS_E$, $\hat{\sigma}^2_{\tau\beta} = (MS_{AB} - MS_E)/n$, $\hat{\sigma}^2_\tau = (MS_A - MS_{AB})/bn$, and $\hat{\sigma}^2_\beta = (MS_B - MS_{AB})/an$.
Example. It is suspected that the strength of a fabric depends on looms (factor A) and operators' experience (factor B). Five looms and four operators are randomly chosen from a large number of looms and operators in a factory. Every operator works on every loom producing three samples of fabric. The data yield $SS_A = 23.6$, $SS_B = 17.8$, $SS_{AB} = 5.6$, and $SS_E = 12.0$.
(a) Analyze the data.
$MS_{AB}/MS_E = 1.56$, $MS_A/MS_{AB} = 12.64$, $MS_B/MS_{AB} = 12.71$, $a = 5$, $b = 4$, $n = 3$, $F_{0.05,\,12,\,40} = 2.00$, $F_{0.05,\,4,\,12} = 3.26$, $F_{0.05,\,3,\,12} = 3.49$
(b) Estimate the variance components.
$\hat{\sigma}^2 = 0.3$, $\hat{\sigma}^2_\tau = 0.45$, $\hat{\sigma}^2_\beta = 0.36$, $\hat{\sigma}^2_{\tau\beta} = 0.06$

Sometimes the estimate of a variance is negative. There are two ways to deal with a negative estimate of a variance component. The first one is to assume that the variance is equal to zero and leave the other estimates unchanged. The second method is to assume the variance is zero and recompute the other estimates using the reduced model.
Example. Suppose $\hat{\sigma}^2_{\tau\beta} = -0.14$. The reduced model is $y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}$, $\mathbb{E}(MS_A) = \sigma^2 + bn\sigma^2_\tau$, $\mathbb{E}(MS_B) = \sigma^2 + an\sigma^2_\beta$, $\hat{\sigma}^2_\tau = (MS_A - MS_E)/bn$, $\hat{\sigma}^2_\beta = (MS_B - MS_E)/an$.

13.3. The Two-Factor Mixed-Effects Model.

Definition. A <u>two-factor mixed-effects model</u> has factor A fixed and factor B random. The interaction term is, thus, also random.
The statistical model is $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$ where
(1) τ_i's are fixed and $\sum_{i=1}^a \tau_i = 0$;
(2) β_j's are i.i.d. $N(0, \sigma^2_\beta)$;
(3) $(\tau\beta)_{ij}$'s are $N(0, \frac{a-1}{a}\sigma^2_{\tau\beta})$ and $\sum_{i=1}^a (\tau\beta)_{ij} = 0$;
(4) $\mathbb{C}ov((\tau\beta)_{ij}, (\tau\beta)_{i'j}) = -\frac{1}{a}\sigma^2_{\tau\beta}$, $i \neq i'$ and $\mathbb{C}ov((\tau\beta)_{ij}, (\tau\beta)_{ij'}) = 0$, $j \neq j'$;
(5) β_j's, $(\tau\beta)_{ij}$'s and ε_{ijk}'s are independent;

(6) ε_{ijk}'s are $N(0, \sigma^2)$.

It can be shown that $\mathbb{E}(MS_A) = \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_{i=1}^a \tau_i^2}{a-1}$, $\mathbb{E}(MS_B) = \sigma^2 + an\sigma^2_\beta$, $\mathbb{E}(MS_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}$, and $\mathbb{E}(MS_E) = \sigma^2$.
The statistic for $H_0 : \tau_i = 0$ for all i is $F = MS_A/MS_{AB}$ which has the F distribution with $a-1$ and $(a-1)(b-1)$ degrees of freedom.
For testing $H_0 : \sigma^2_\beta = 0$ the test statistic is $F = MS_B/MS_E$ which has the F distribution with $b-1$ and $ab(n-1)$ degrees of freedom.
Thus, the test statistic for $H_0 : \sigma^2_{\tau\beta} = 0$ is $F = MS_{AB}/MS_E$ which has the F distribution with $(a-1)(b-1)$ and $ab(n-1)$ degrees of freedom.
The estimators of the fixed factor effects are $\hat{\mu} = \bar{y}_{...}$ and $\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$. The variance components are estimated by $\hat{\sigma}^2 = MS_E$, $\hat{\sigma}^2_\beta = (MS_B - MS_E)/an$ and $\hat{\sigma}^2_{\tau\beta} = (MS_{AB} - MS_E)/n$.
Example. Experimenters wish to know whether there is a difference in performance of a certain heart valve by center and by size. Three centers are randomly chosen from a long list of sites where the valve is implanted. The five valve sizes are 19mm, 21mm, 23mm, 25mm, and 27mm. Two patients for each size are implanted at each center and their survival times are recorded. The data are summarized as $SS_A = 2.13,\ SS_B = 4.12,\ SS_{AB} = 9.87,\ SS_E = 7.45,\ y_{...} = 294,\ y_{1..} = 56,\ y_{2..} = 88,\ y_{3..} = 72,\ y_{4..} = 34,\ y_{5..} = 44$.
(a) Analyze the data.
$MS_A = 0.53,\ MS_B = 2.06,\ MS_{AB} = 1.23,\ MS_E = 0.5,\ MS_A/MS_{AB} = 0.43,\ MS_B/MS_E = 4.12,\ MS_{AB}/MS_E = 2.46,\ a = 5,\ b = 3,\ n = 2,\ F_{0.05,4,8} = 3.84,\ F_{0.05,2,15} = 3.68,\ F_{0.05,8,15} = 2.64$
(b) Estimate the parameters of the model.
$\hat{\mu} = 9.8,\ \hat{\tau}_1 = -0.47,\ \hat{\tau}_2 = 4.87,\ \hat{\tau}_3 = 2.2,\ \hat{\tau}_4 = -4.13,\ \hat{\tau}_5 = -2.47,\ \hat{\sigma}^2 = 0.5,\ \hat{\sigma}^2_\beta = 0.156,\ \hat{\sigma}^2_{\tau\beta} = 0.365$

14.1. The Two-Stage Nested Design.

Example. A study is performed to compare the reliability of several models of cars. Two different car models from each of three domestic car manufacturers are tested. Two different cars of each make and model are subjected to a reliability test. The scores are recorded. The data are

Make 1		Make 2		Make 3	
Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
62	77	72	58	94	85
67	73	69	57	90	88

This is a nested design.
Definition. A two-stage nested design (or hierarchical design) has two factors A and B, with the levels of factor B being similar but not identical for different levels of factor A. Factor B is said to be nested under the levels of factor A.
In our example, the factor "Model" is nested under the factor "Make." The two Models are not the same for different Makes, and could be potentially

numbered 1 through 6.

Statistical Analysis.

The model is $y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{(ij)k}$, $i = 1, \dots, a$, $j = 1, \dots, b$, $k = 1, \dots, n$ where $\beta_{j(i)}$ is the effect of the jth level of B nested under the ith level of A. The ANOVA table is

Source	SS	DF
A	$\frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn}$	$a-1$
B within A	$\frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2$	$a(b-1)$
Error	by subtraction	$ab(n-1)$
Total	SS_T	$abn-1$

Here are the expected values of the mean squares depending on whether factors are fixed or random.

$\mathbb{E}(MS)$	A Fixed B Fixed	A Fixed B Random	A Random B Random
$\mathbb{E}(MS_A)$	$\sigma^2 + \frac{bn\sum\tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + \frac{bn\sum\tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$\mathbb{E}(MS_{B(A)})$	$\sigma^2 + \frac{n\sum\sum\beta_{j(i)}^2}{a(b-1)}$	$\sigma^2 + n\sigma_\beta^2$	$\sigma^2 + n\sigma_\beta^2$
$\mathbb{E}(MS_E)$	σ^2	σ^2	σ^2

Case 1. A fixed, B fixed. The model constraints are $\sum_{i=1}^{a}\tau_i = 0$ and $\sum_{j=1}^{b}\beta_{j(i)} = 0$, $i = 1, \dots, a$. The hypothesis $H_0 : \tau_i = 0$ is tested by MS_A/MS_E, and $H_0 : \beta_{j(i)} = 0$ is tested by $MS_{B(A)}/MS_E$.
Case 2. A fixed, B random. The model assumptions are $\sum_{i=1}^{a}\tau_i = 0$ and $\beta_{j(i)}$ are i.i.d. $N(0, \sigma_\beta^2)$. Test statistic for $H_0 : \tau_i = 0$ is $MS_A/MS_{B(A)}$ and that for $H_0 : \sigma_\beta^2 = 0$ is $MS_{B(A)}/MS_E$.
Case 3. A random, B random. Model assumptions are τ_i are i.i.d. $N(0, \sigma_\tau^2)$ and $\beta_{j(i)}$ are i.i.d. $N(0, \sigma_\beta^2)$. Test statistic for $H_0 : \sigma_\tau^2 = 0$ is $MS_A/MS_{B(A)}$ and that for $H_0 : \sigma_\beta^2 = 0$ is $MS_{B(A)}/MS_E$.

Example. In our example, both A and B are fixed. The ANOVA table is

Source	SS	DF	MS	F
Make	1401.17	2	700.58	110.68
Model (within Make)	309.5	3	103.17	16.29
Error	38	6	6.33	
Total	1748.67	11		

15.3. The Analysis of Covariance (ANCOVA).

Definition. ANCOVA is an experimental design in the presence of an independent variable (called covariate).

Statistical Model The model is $y_{ij} = \mu + \tau_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$, where $i = 1, \ldots, a,\ j = 1, \ldots, n,\ \varepsilon_{ij} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, and $\sum_{i=1}^{a} \tau_i = 0$. To obtain the least-squares estimates of the parameters μ, β, and τ_i, minimize the sum of squares

$$\sum_{i=1}^{a} \sum_{j=1}^{n} \big(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}(x_{ij} - \bar{x}_{..}) \big)^2$$

under the assumption that $\sum_{i=1}^{a} \hat{\tau}_i = 0$. Solving the system of normal equations for $\hat{\mu}$ and $\hat{\tau}_i$, arrive at the estimates

$$\hat{\mu} = \bar{y}_{..}, \quad \text{and} \quad \hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}),$$

Substituting $\hat{\tau}_i$ into the normal equation for $\hat{\beta}$ yields

$$\hat{\beta} \sum_{i=1}^{a} \sum_{j=1}^{n} \big[(x_{ij} - \bar{x}_{..})^2 - (\bar{x}_{i.} - \bar{x}_{..})(x_{ij} - \bar{x}_{..}) \big] = \sum_{i=1}^{a} \sum_{j=1}^{n} \big[y_{ij}(x_{ij} - \bar{x}_{..}) - \bar{y}_{i.}(x_{ij} - \bar{x}_{..}) \big]$$

or

$$\hat{\beta} = \frac{S_{xy} - T_{xy}}{S_{xx} - T_{xx}} = \frac{E_{xy}}{E_{xx}}$$

where

$$S_{xx} = \sum_{i=1}^{a} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{..})^2,$$

$$S_{xy} = \sum_{i=1}^{a} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..}) = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}\,(x_{ij} - \bar{x}_{..}),$$

$$T_{xx} = \sum_{i=1}^{a} \sum_{j=1}^{n} (\bar{x}_{i.} - \bar{x}_{..})(x_{ij} - \bar{x}_{..}) = n \sum_{i=1}^{a} (\bar{x}_{i.} - \bar{x}_{..})^2,$$

$$T_{xy} = \sum_{i=1}^{a} \sum_{j=1}^{n} (\bar{x}_{i.} - \bar{x}_{..})(y_{ij} - \bar{y}_{..}) = \sum_{i=1}^{a} \sum_{j=1}^{n} \bar{y}_{i.}(x_{ij} - \bar{x}_{..}) = n \sum_{i=1}^{a} (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..}).$$

The ANCOVA table.

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{an}, \quad SS_{tr} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{an},$$

$$SS_E = SS_T - SS_{tr}.$$

The sums of squares adjusted to the presence of the covariate are

$$SS_{T(adj)} = SS_T - \frac{S_{xy}^2}{S_{xx}}, \quad df_{T(adj)} = an - 2,$$

$$SS_{E(adj)} = SS_E - \frac{E_{xy}^2}{E_{xx}}, \quad df_{E(adj)} = a(n-1) - 1,$$

$$SS_{tr(adj)} = SS_{T(adj)} - SS_{E(adj)}, \quad df_{tr(adj)} = a - 1.$$

The test statistic for testing $H_0 : \tau_1 = \tau_2 = \cdots = \tau_a = 0$ againts $H_1 : \tau_i \neq 0$ for some i is

$$F = \frac{SS_{tr(adj)}/(a-1)}{SS_{E(adj)}/(a(n-1)-1)} \sim F\big(a-1, a(n-1)-1)\big).$$

To test the hypotheses $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$, compute

$$SS_{slope} = \frac{E_{xy}^2}{E_{xx}}, \quad df_{slope} = 1,$$

$$F = \frac{SS_{slope}/1}{SS_{E(adj)}/(a(n-1)-1)}.$$

The estimator of σ^2 is $\hat{\sigma}^2 = MS_{E(adj)}$.

Example. Experimental units = 12 steel brackets, treatments = 3 shops where steel brackets are zinc plated, response y= thickness of zinc plating, covariate x= thickness of steel bracket before plating. The data are

Shop					
1		2		3	
x	y	x	y	x	y
110	40	60	25	62	27
75	38	75	32	90	24
93	30	38	13	45	20
98	47	140	35	59	13

The model is $y_{ij} = \mu + \tau_i + \beta(x_{ij} - \bar{x}_{..}) + \varepsilon_{ij}$, where $i = 1, \dots, 3$, $j = 1, \dots, 4$, $\varepsilon_{ij} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, and $\sum_{i=1}^{a} \tau_i = 0$. Compute $\bar{x}_{1.} = 94$, $\bar{y}_{1.} = 38.75$, $\bar{x}_{2.} = 78.25$, $\bar{y}_{2.} = 26.25$, $\bar{x}_{3.} = 64$, $\bar{y}_{3.} = 21$, $\bar{x}_{..} = 78.75$, $\bar{y}_{..} = 28.6667 = \hat{\mu}$.

$S_{xx} = 9278.25,\ S_{xy} = 2351,\ T_{xx} = 1801.5,\ T_{xy} = 1072.25,\ E_{xy} = 1278.75,\ E_{xx} = 7476.75,\ \hat{\beta} = 0.17,\ \hat{\tau}_1 = 7.48,\ \hat{\tau}_2 = -2.33,\ \hat{\tau}_3 = -5.14.\ SS_T = 1208.67,\ SS_{tr} = 665.17,\ SS_E = 543.5,\ SS_{T(adj)} = 612.95,\ SS_{E(adj)} = 324.80, SS_{tr(adj)} = 288.15,\ SS_{slope} = 218.70,\ \hat{\sigma}^2 = 40.60.$
The ANCOVA table is

Source	SS	DF	MS	F	F_{cr}
Treatment(adj)	288.15	2	144.08	3.55 $<$	4.46
(Slope)	(218.70)	(1)	(218.70)	(5.39) $>$	5.32
Error(adj)	324.80	8	40.60		
Total(adj)	612.95	10			

The conclusion of the ANCOVA analysis is that there is no difference between the shops but the presence of the covariate in the model is significant.

If we ignore the covariate, then the ANOVA table is

Source	SS	DF	MS	F	F_{cr}
Treatment	665.17	2	332.58	5.507 $>$	4.26
Error	543.5	9	60.39		
Total	1208.67	11			

The conclusion of the ANOVA analysis is that the shops differ significantly.

The Cross-Over Design.

DEFINITION. Cross-over design is a design in which several treatments are applied sequentially to the same experimental unit. The advantage of this design is that treatments are compared on the same unit, and therefore, there is no between-unit variability. The disadvantage is that there might be a carry-over (or residual) effect: a treatment may have an effect beyond the period during which it is applied.
As an example, we will consider the simplest case of a cross-over design.

SIMPLE REVERSAL (AB/BA) CROSS-OVER DESIGN

In this design, half of the experimental units receive treatment A, then, after a wash-out period, treatment B; the other half of the units receive treatment B first, and then, after the wash-out period, treatment A.

The collected data are of the form

Period	Treatment	Group 1				Total
1	A	y_{111}	y_{121}	$\cdots$	y_{1n_11}	$y_{1\cdot 1}$
2	B	y_{112}	y_{122}	$\cdots$	y_{1n_12}	$y_{1\cdot 2}$
	Difference	D_{11}	D_{12}	$\dots$	D_{1n_1}	G_1

Period	Treatment	Group 2				Total
1	B	y_{211}	y_{221}	$\cdots$	y_{2n_21}	$y_{2\cdot 1}$
2	A	y_{212}	y_{222}	$\cdots$	y_{2n_22}	$y_{2\cdot 2}$
	Difference	D_{21}	D_{22}	$\dots$	D_{2n_2}	G_2

Here y_{ijk} is an observation on the j-th subject from the i-th group, at period k, $i = 1$ or 2, $j = 1, \dots, n_i$, and $k = 1$ or 2. The difference $D_{ij} = y_{ij\,1} - y_{ij\,2}$, and the grand differences $G_1 = y_{1\cdot 1} - y_{1\cdot 2}$ and $G_2 = y_{2\cdot 1} - y_{2\cdot 2}$.

The model is more conveniently written in the form:

$$y_{ijkl} = \mu + \alpha_i + \beta_{ij} + \gamma_k + \tau_l + \varepsilon_{ijkl},$$

where μ is the overall mean, α_i is the i-th group effect, $i = 1$ or 2, β_{ij} is the effect of the j-th subject from group i, $j = 1, \dots, n_i$, $\beta_{ij} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2_\beta)$, γ_k is the k-th period effect, $k = 1$ or 2, τ_l is the l-th treatment effect, $l = 1$ or 2 depending on i and k, and $\varepsilon_{ijkl} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$. The parameters of this model are μ, α_i, σ^2_β, γ_k, τ_l and σ^2. We are interested in testing $H_0 : \tau_1 = \tau_2 = 0$ against $H_1 : \tau_1 \neq 0$ and $\tau_2 \neq 0$.

The analysis of the data is as follows:

$$SS_{tr} = \frac{(n_2G_1 - n_1G_2)^2}{2n_1n_2(n_1 + n_2)}, \;\; df_{tr} = 1,$$

$$SS_E = \frac{1}{2}\Big(\sum_{j=1}^{n_1} D^2_{1j} + \sum_{j=1}^{n_2} D^2_{2j}\Big) - \frac{1}{2n_1}\,G_1^2 - \frac{1}{2n_2}\,G_2^2, \;\; df_E = n_1 + n_2 - 2,$$

$$F = \frac{MS_{tr}}{MS_E} \sim F(1, n_1 + n_2 - 2).$$

When $n_1 = n_2 = n$, the formulas simplify to become

$$SS_{tr} = \frac{(G_1 - G_2)^2}{4n}, \;\; df_{tr} = 1,$$

and

$$SS_E = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{n} D^2_{ij} - \frac{1}{2n}\left(G_1^2 + G_2^2\right), \;\; df_E = 2(n - 1).$$

Example. Two drugs A or B are applied to subjects and the reaction time (in seconds) to certain stimulus is measured. There are inherent differences in reaction time among individuals, therefore, a cross-over design is used with a one-week wash-out period. The data are

Group 1		Subject				
Period	Drug	1	2	3	4	Total
1	A	30	57	52	66	205
2	B	28	50	38	49	165
	Difference	2	7	14	17	40

Group 2		Subject				
Period	Drug	1	2	3	4	Total
1	B	34	32	18	28	112
2	A	41	21	27	36	125
	Difference	-7	11	-9	-8	-13

The ANOVA table is

Source	SS	DF	MS	F	$F_{0.05,1,6}$
Treatment	175.5625	1	175.5625	5.129 $<$	5.987
Error	205.375	6	34.229		

The conclusion is that there is no statistically significant difference between the two drugs.

3.3.2. Fixed Effects Model: Statistical Analysis.
SAS Code

```
title 'The Difference in Fabric Strength for the Three Looms';
data textile;
  input loom strength;
cards;
1 88
1 93
1 90
1 89
2 91
2 89
2 92
2 94
3 97
3 96
3 94
3 93
;

proc anova data=textile;
        class loom;
        model strength = loom;
  run;
```

- -

SAS Output

The Difference in Fabric Strength for the Three Looms

The ANOVA Procedure

Dependent Variable: strength

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	52.66666667	26.33333333	6.41	0.0186
Error	9	37.00000000	4.11111111		
Corrected Total	11	89.66666667			

3.5.7. Comparing Pairs of Treatment Means.

SAS Code

```
title 'The Difference in Fabric Strength for the Three Looms';
data textile;
  input loom strength;
cards;
1 88
1 93
1 90
1 89
2 91
2 89
2 92
2 94
3 97
3 96
3 94
3 93
;

proc anova data=textile;
        class loom;
        model strength = loom;
        means loom / lsd clm cldiff;
  run;


proc anova data=textile;
        class loom;
        model strength = loom;
        means loom / tukey lsd duncan;
  run;
```

- -

SAS Output

loom	N	Mean	95% Confidence Limits	
3	4	95.000	92.707	97.293
2	4	91.500	89.207	93.793
1	4	90.000	87.707	92.293

The Difference in Fabric Strength for the Three Looms

loom Comparison	Difference Between Means	95% Confidence Limits		
3 - 2	3.500	0.257	6.743	***
3 - 1	5.000	1.757	8.243	***
2 - 3	-3.500	-6.743	-0.257	***
2 - 1	1.500	-1.743	4.743	
1 - 3	-5.000	-8.243	-1.757	***
1 - 2	-1.500	-4.743	1.743	

Tukey's Studentized Range (HSD) Test for strength

Alpha	0.05
Error Degrees of Freedom	9
Error Mean Square	4.111111
Critical Value of Studentized Range	3.94850
Minimum Significant Difference	4.003

Means with the same letter are not significantly different.

Tukey Grouping		Mean	N	loom
	A	95.000	4	3
	A			
B	A	91.500	4	2
B				
B		90.000	4	1

t Tests (LSD) for strength

Alpha	0.05
Error Degrees of Freedom	9
Error Mean Square	4.111111
Critical Value of t	2.26216
Least Significant Difference	3.2433

Means with the same letter are not significantly different.

t Grouping	Mean	N	loom
A	95.000	4	3
B	91.500	4	2
B			
B	90.000	4	1

Duncan's Multiple Range Test for strength

Alpha	0.05
Error Degrees of Freedom	9
Error Mean Square	4.111111

Number of Means	2	3
Critical Range	3.243	3.385

Means with the same letter are not significantly different.

Duncan Grouping	Mean	N	loom
A	95.000	4	3
B	91.500	4	2
B			
B	90.000	4	1

3.5.5. Orthogonal Contrasts.
SAS Code

```
title 'Orthogonal contrasts';
data textile;
  input loom strength @@;
datalines;
1 88 1 93 1 90 1 89
2 91 2 89 2 92 2 94
3 97 3 96 3 94 3 93
;

proc glm data=textile;
        class loom;
        model strength=loom;
            means loom;
              contrast ‘Contrast 1’ loom 1 0 -1;
              contrast ‘Contrast 2’ loom 1 -2 1;
run;
```

- -

SAS Output

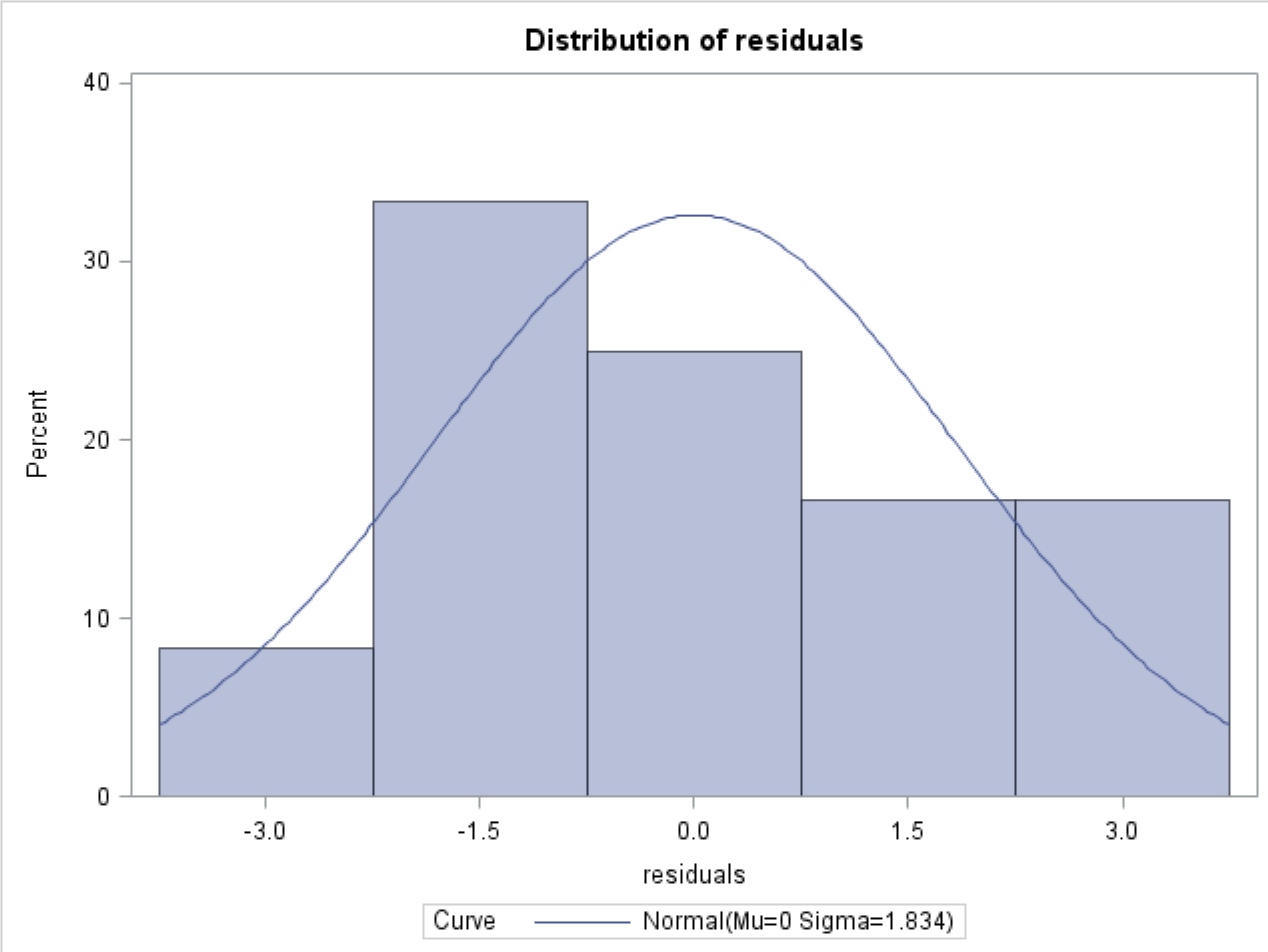
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	52.66666667	26.33333333	6.41	0.0186
Error	9	37.00000000	4.11111111		
Corrected Total	11	89.66666667			

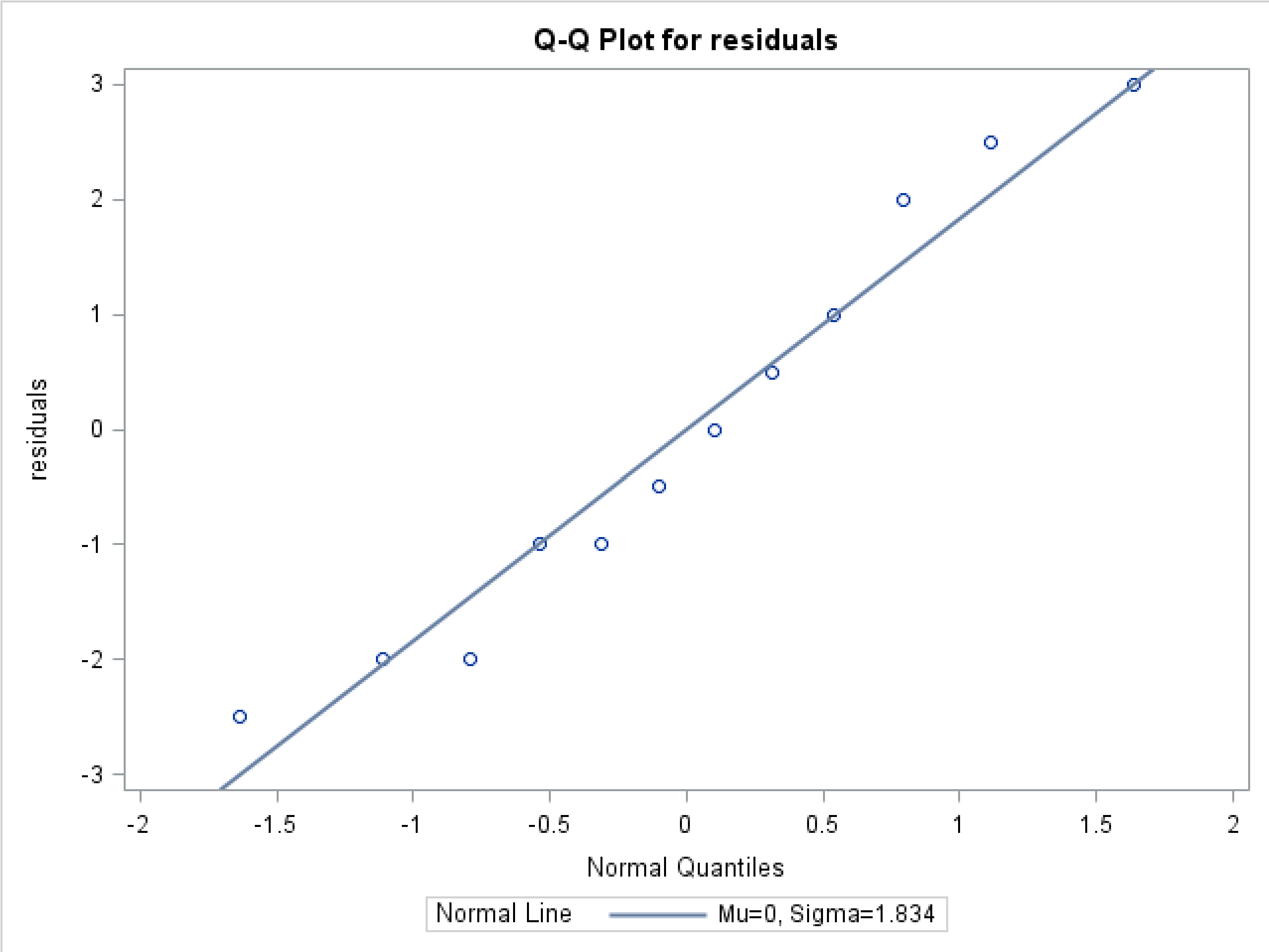
Contrast 1	1	50.00000000	50.00000000	12.16	0.0069
Contrast 2	1	2.66666667	2.66666667	0.65	0.4414

3.4. Model Adequacy Checking.
SAS Code

```
title 'The Difference in Fabric Strength for the Three Looms';
data textile;
  input loom strength;
cards;
1 88
1 93
1 90
1 89
2 91
2 89
2 92
2 94
3 97
3 96
3 94
3 93
;

proc glm data=textile;
        class loom;
        model strength=loom;
            means loom/ hovtest=bartlett;
            output out=new r=residuals p=pstrength;
     run;


  proc capability normaltest data=new;
        var residuals;
        histogram residuals/normal;
        qqplot residuals/normal (mu=est sigma=est);
     run;


  proc gplot data=new;
         plot residuals*pstrength/vref=0;
     run;
```

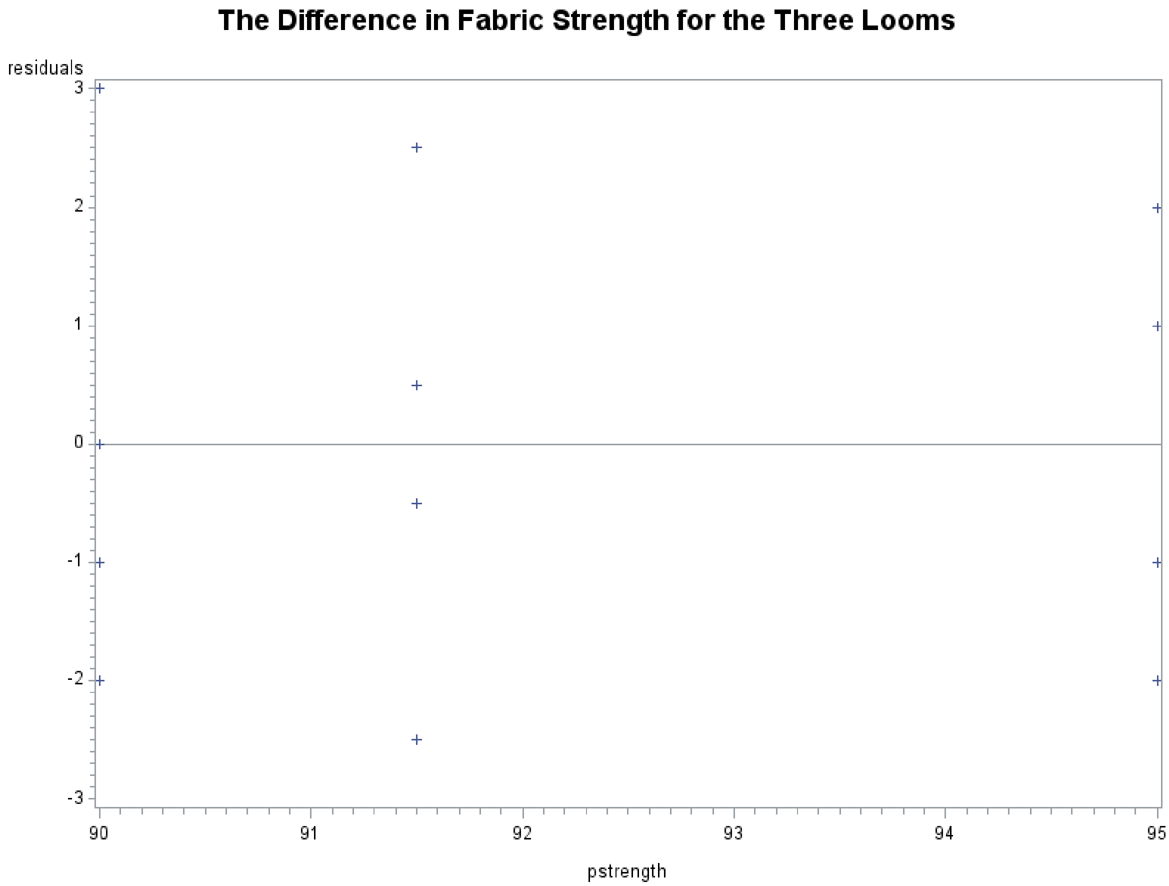
SAS Output


Distribution of residuals
Percent
40
30
20
10
0
-3.0
-1.5
0.0
1.5
3.0
residuals
Curve
Normal(Mu=0 Sigma=1.834)



Goodness-of-Fit Tests for Normal Distribution

Test		Statistic	DF	p Value	
Kolmogorov-Smirnov	**D**	0.12387579		**Pr > D**	>0.150
Cramer-von Mises	**W-Sq**	0.03377973		**Pr > W-Sq**	>0.250
Anderson-Darling	**A-Sq**	0.24883072		**Pr > A-Sq**	>0.250
Chi-Square	**Chi-Sq**	1.73172167	2	**Pr > Chi-Sq**	0.421



Bartlett's Test for Homogeneity of strength Variance			
Source	**DF**	**Chi-Square**	**Pr > ChiSq**
loom	2	0.0792	0.9612

4.1. The Randomized Complete Block Design.
SAS Code

```
data RCBD;
   input loom operator strength @@;
datalines;
1 1 88 1 2 93 1 3 90
2 1 90 2 2 92 2 3 92
3 1 91 3 2 96 3 3 94
;

proc anova;
        class loom operator;
        model strength = loom operator;
  run;
```

- -

SAS Output

Source	DF	Squares	Mean Square	F Value	Pr > F
Model	4	41.77777778	10.44444444	11.06	0.0195
Error	4	3.77777778	0.94444444		
Corrected Total	8	45.55555556			

R-Square	Coeff Var	Root MSE	strength Mean
0.917073	1.058890	0.971825	91.77778

Source	DF	Anova SS	Mean Square	F Value	Pr > F
loom	2	17.55555556	8.77777778	9.29	0.0314
operator	2	24.22222222	12.11111111	12.82	0.0182

4.2. The Latin Square Design.
SAS Code

```
data LATIN_SQUARE;
  input batch operator formulation $ force @@;
datalines;
1 1 A -1 1 2 B -5 1 3 C -6 1 4 D -1 1 5 E -1
2 1 B -8 2 2 C -1 2 3 D 5 2 4 E 2 2 5 A 11
3 1 C -7 3 2 D 13 3 3 E 1 3 4 A 2 3 5 B -4
4 1 D 1 4 2 E 6 4 3 A 1 4 4 B -2 4 5 C -3
5 1 E -3 5 2 A 5 5 3 B -5 5 4 C 4 5 5 D 6
;

proc anova;
        class batch operator formulation;
        model force = batch operator formulation;
  run;
```

- -

SAS Output

Source	DF	Squares	Mean Square	F Value	Pr > F
Model	12	548.0000000	45.6666667	4.28	0.0089
Error	12	128.0000000	10.6666667		
Corrected Total	24	676.0000000			

R-Square	Coeff Var	Root MSE	force Mean
0.810651	816.4966	3.265986	0.400000

Source	DF	Anova SS	Mean Square	F Value	Pr > F
batch	4	68.0000000	17.0000000	1.59	0.2391
operator	4	150.0000000	37.5000000	3.52	0.0404
formulation	4	330.0000000	82.5000000	7.73	0.0025

4.3. The Græco-Latin Square Design.
SAS Code

```
data GRAECO_LATIN;
  input batch operator formulation $ assembly $ force @@;
datalines;
1 1 A a -1 1 2 B c -5 1 3 C e -6 1 4 D b -1 1 5 E d -1
2 1 B b -8 2 2 C d -1 2 3 D a 5 2 4 E c 2 2 5 A e 11
3 1 C c -7 3 2 D e 13 3 3 E b 1 3 4 A d 2 3 5 B a -4
4 1 D d 1 4 2 E a 6 4 3 A c 1 4 4 B e -2 4 5 C b -3
5 1 E e -3 5 2 A b 5 5 3 B d -5 5 4 C a 4 5 5 D c 6
;


proc anova;
       class batch operator formulation assembly;
       model force = batch operator formulation assembly;
  run;
```

- -

SAS Output

Source	DF	Squares	Mean Square	F Value	Pr > F
Model	16	610.0000000	38.1250000	4.62	0.0171
Error	8	66.0000000	8.2500000		
Corrected Total	24	676.0000000			

R-Square	Coeff Var	Root MSE	force Mean
0.902367	718.0703	2.872281	0.400000

Source	DF	Anova SS	Mean Square	F Value	Pr > F
batch	4	68.0000000	17.0000000	2.06	0.1783
operator	4	150.0000000	37.5000000	4.55	0.0329
formulation	4	330.0000000	82.5000000	10.00	0.0033
assembly	4	62.0000000	15.5000000	1.88	0.2076

4.4. The Balanced Incomplete Block Design.

SAS Code

```
data BIBD;
  input catalyst batch time @@;
datalines;
1 1 73 1 2 74 1 4 71
2 2 75 2 3 67 2 4 72
3 1 73 3 2 75 3 3 68
4 1 75 4 3 72 4 4 75
;


proc glm;
        class catalyst batch;
        model time=batch catalyst;
  run;
```

- -

SAS Output

Source	DF	Squares	Mean Square	F Value	Pr > F
Model	6	77.75000000	12.95833333	19.94	0.0024
Error	5	3.25000000	0.65000000		
Corrected Total	11	81.00000000			

R-Square	Coeff Var	Root MSE	time Mean
0.959877	1.112036	0.806226	72.50000

Source	DF	Type I SS	Mean Square	F Value	Pr > F
batch	3	55.00000000	18.33333333	28.21	0.0015
catalyst	3	22.75000000	7.58333333	11.67	0.0107

Source	DF	Type III SS	Mean Square	F Value	Pr > F
batch	3	66.08333333	22.02777778	33.89	0.0010
catalyst	3	22.75000000	7.58333333	11.67	0.0107

5.3.2. The Two-Factor Factorial Design: Fixed-Effects Model. SAS Code

```
data factorial_design;
  input  temp conc yield @@;
datalines;
1 1 17 1 1 20 1 2 16 1 2 21 1 3 24 1 3 22
2 1 12 2 1 9 2 2 18 2 2 13 2 3 17 2 3 12
3 1 16 3 1 12 3 2 18 3 2 21 3 3 25 3 3 23
;

proc glm;
        class temp conc;
        model yield=temp conc temp*conc;
            lsmeans temp conc/out=maineff;
            lsmeans temp*conc/out=intereff;
     run;

symbol interpol=join value=x;

proc gplot data=maineff;
   plot lsmean*temp;
   plot lsmean*conc;
run;

proc gplot data=intereff;
   plot lsmean*conc=temp;
run;
```

- -

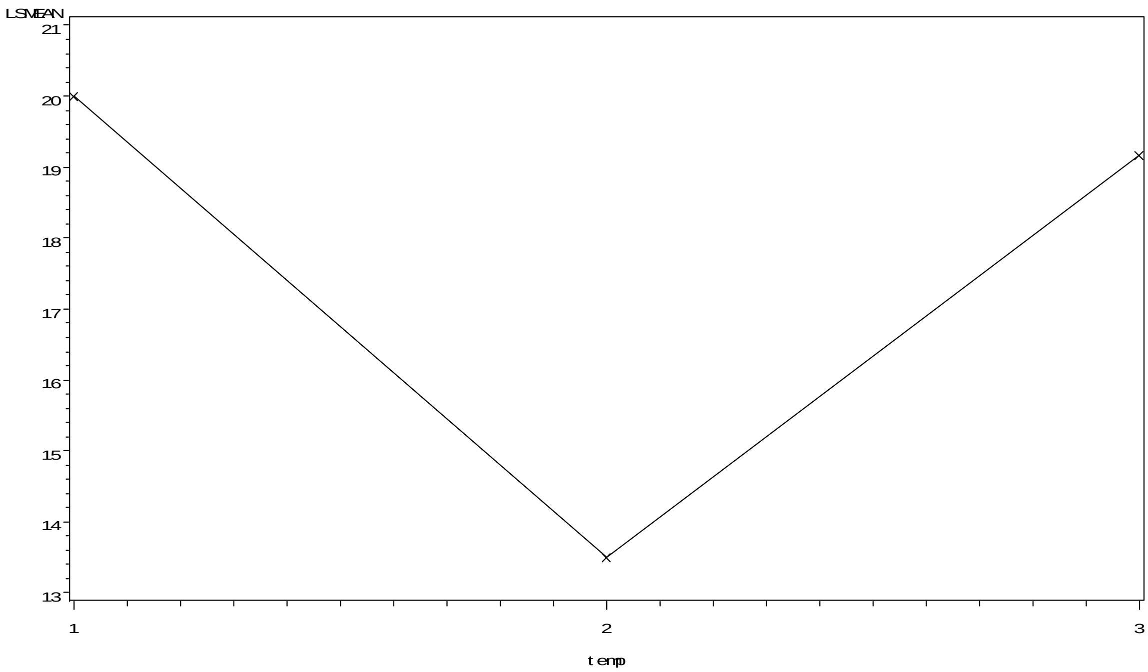
SAS Output

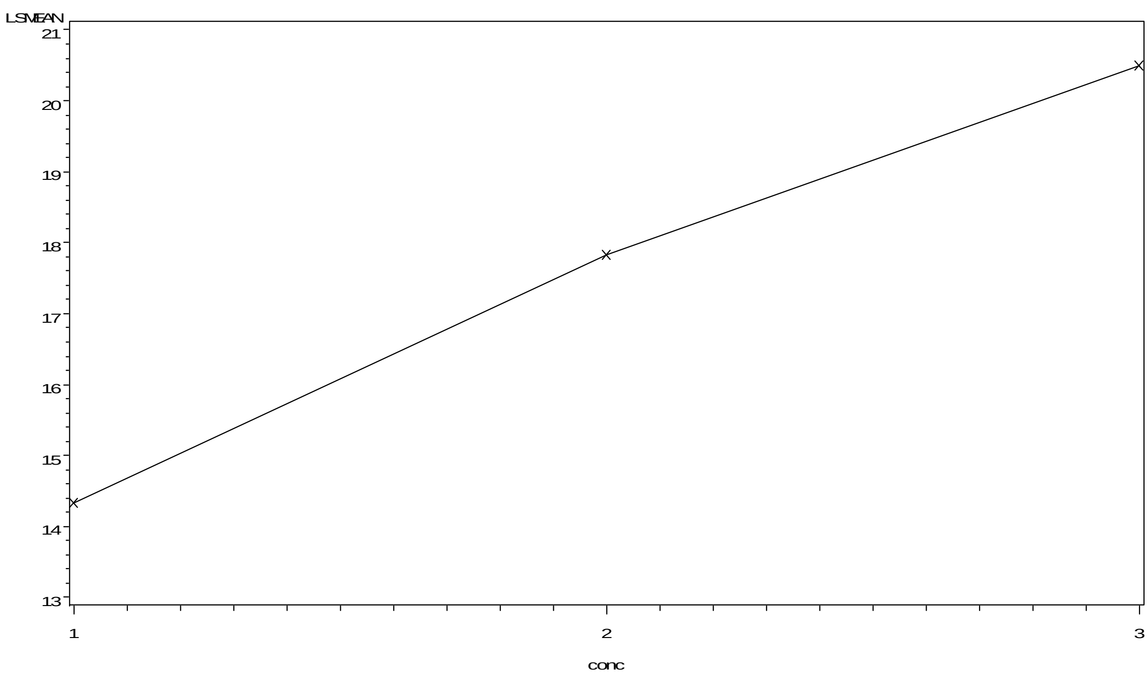
```
Error                9      63.000      63.000      7.000
Total               17     368.444
```

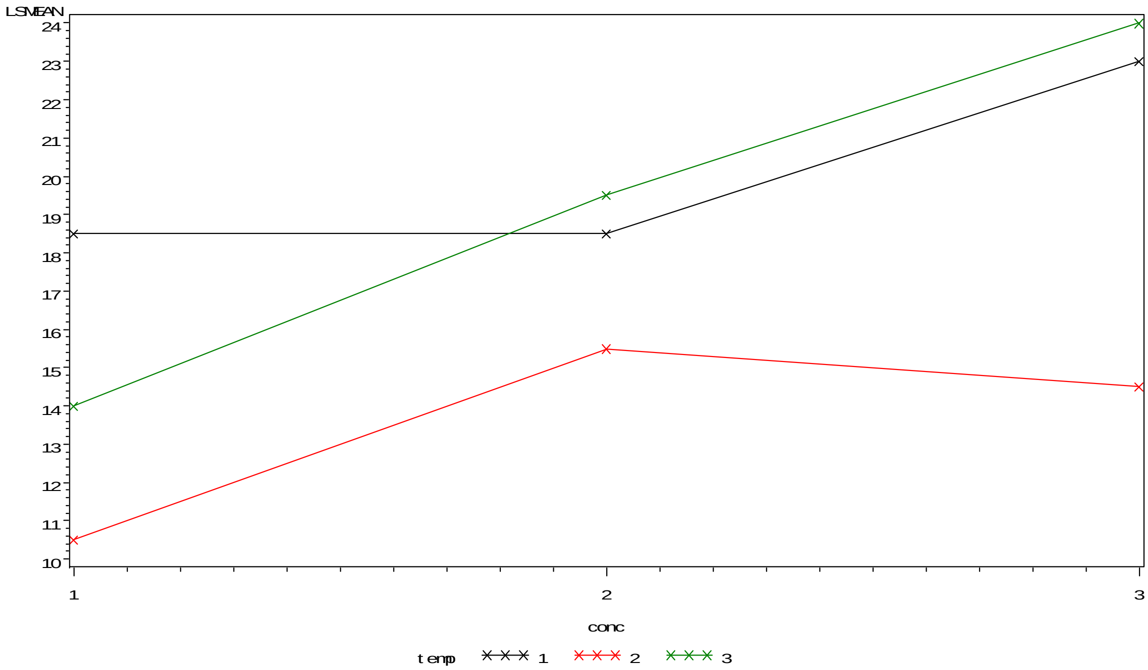
Source	DF	Squares	Mean Square	F Value	Pr > F
Model	8	305.4444444	38.1805556	5.45	0.0101
Error	9	63.0000000	7.0000000		
Corrected Total	17	368.4444444			

R-Square	Coeff Var	Root MSE	yield Mean
0.829011	15.07074	2.645751	17.55556

Source	DF	Type I SS	Mean Square	F Value	Pr > F
temp	2	150.1111111	75.0555556	10.72	0.0042
conc	2	114.7777778	57.3888889	8.20	0.0094
temp*conc	4	40.5555556	10.1388889	1.45	0.2951


LSMEAN
21
20
19
18
17
16
15
14
13
1
2
3
temp


LSMEAN
21
20
19
18
17
16
15
14
13
1
2
3
conc


LSMEAN
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
1
2
3
conc
temp
1
2
3

6.3. The 2^3 Factorial Design.

SAS Code

```
data two_cubed;
  input  temp conc catalyst yield @@;
datalines;
1 1 1 18.2 1 1 1 18.9 2 1 1 12.9 2 1 1 14.4
1 2 1 27.2 1 2 1 24.0 2 2 1 22.4 2 2 1 22.5
1 1 2 15.9 1 1 2 14.5 2 1 2 15.1 2 1 2 14.2
1 2 2 41.0 1 2 2 43.0 2 2 2 36.3 2 2 2 39.9
;

proc glm;
        class temp conc catalyst;
        model yield=temp | conc | catalyst;
     run;
```

SAS Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	1658.110000	236.872857	115.83	<.0001
Error	8	16.360000	2.045000		
Corrected Total	15	1674.470000			

R-Square	Coeff Var	Root MSE	yield Mean
0.990230	6.014868	1.430035	23.77500

Source	DF	Type I SS	Mean Square	F Value	Pr > F
temp	1	39.062500	39.062500	19.10	0.0024
conc	1	1092.302500	1092.302500	534.13	<.0001
temp*conc	1	0.640000	0.640000	0.31	0.5912
catalyst	1	220.522500	220.522500	107.83	<.0001
temp*catalyst	1	3.240000	3.240000	1.58	0.2436
conc*catalyst	1	295.840000	295.840000	144.67	<.0001
temp*conc*catalyst	1	6.502500	6.502500	3.18	0.1124

5.6. Blocking in a Factorial Design.
SAS Code

```
data one;
input temp day conc yield @@;
datalines;
1 1 1 17 1 1 2 16 1 1 3 24 1 2 1 20 1 2 2 21 1 2 3 22
2 1 1 12 2 1 2 18 2 1 3 17 2 2 1  9 2 2 2 13 2 2 3 12
3 1 1 16 3 1 2 18 3 1 3 25 3 2 1 12 3 2 2 21 3 2 3 23
;

proc glm;
 class temp day conc;
   model yield=temp conc temp*conc day;
run;
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
```

SAS Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	9	311.0000000	34.5555556	4.81	0.0187
Error	8	57.4444444	7.1805556		
Corrected Total	17	368.4444444			

R-Square	Coeff Var	Root MSE	yield Mean
0.844089	15.26386	2.679656	17.55556

Source	DF	Type I SS	Mean Square	F Value	Pr > F
temp	2	150.1111111	75.0555556	10.45	0.0059
conc	2	114.7777778	57.3888889	7.99	0.0124
temp*conc	4	40.5555556	10.1388889	1.41	0.3135
day	1	5.5555556	5.5555556	0.77	0.4047

7.4. Confounding in the 2^k Design in Two Blocks.
SAS Code

```
data one;
input temp conc batch yield @@;
datalines;
1 1 1 28 1 1 3 25 1 1 5 27
2 1 2 36 2 1 4 32 2 1 6 32
2 2 1 31 2 2 3 30 2 2 5 29
1 2 2 18 1 2 4 19 1 2 6 23
;

proc glm;
class temp conc batch;
model yield = temp conc batch;
run;
```

- -

SAS Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	300.3333333	42.9047619	7.57	0.0344
Error	4	22.6666667	5.6666667		
Corrected Total	11	323.0000000			

R-Square	Coeff Var	Root MSE	yield Mean
0.929825	8.656277	2.380476	27.50000

Source	DF	Type I SS	Mean Square	F Value	Pr > F
temp	1	208.3333333	208.3333333	36.76	0.0037
conc	1	75.0000000	75.0000000	13.24	0.0220
batch	5	17.0000000	3.4000000	0.60	0.7080

13.2. The Two-Way ANOVA Design with Random Effects.
SAS Code

```
data random_effects;
  input temp conc yield @@;
datalines;
1 1 17 1 1 20 1 2 16 1 2 21 1 3 24 1 3 22
2 1 12 2 1 9 2 2 18 2 2 13 2 3 17 2 3 12
3 1 16 3 1 12 3 2 18 3 2 21 3 3 25 3 3 23
;

proc glm;
        class temp conc;
        model yield=temp conc temp*conc;
        random temp conc temp*conc/test;
run;
```

- -

SAS Output

```
                                       The GLM Procedure
                     Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: yield

        Source                     DF      Type III SS     Mean Square    F Value     Pr >
F

        temp                        2       150.111111       75.055556       7.40
0.0452
        conc                        2       114.777778       57.388889       5.66
0.0682

        Error: MS(temp*conc)        4        40.555556       10.138889


        Source                     DF      Type III SS     Mean Square    F Value     Pr >
F

        temp*conc                   4        40.555556       10.138889       1.45
0.2951

        Error: MS(Error)            9        63.000000        7.000000
```

13.3. The Two-Way ANOVA Design with Mixed Effects.
SAS Code

```
data mixed_effects;
  input temp conc yield @@;
datalines;
1 1 17 1 1 20 1 2 16 1 2 21 1 3 24 1 3 22
2 1 12 2 1 9 2 2 18 2 2 13 2 3 17 2 3 12
3 1 16 3 1 12 3 2 18 3 2 21 3 3 25 3 3 23
;

proc glm;
        class temp conc;
        model yield=temp conc temp*conc;
            random conc temp*conc / test;

run;
```

- -

SAS Output

The GLM Procedure
Tests of Hypotheses for Mixed Model Analysis of Variance

Dependent Variable: yield

Source	DF	Type III SS	Mean Square	F Value	Pr > F
temp	2	150.111111	75.055556	7.40	0.0452
conc	2	114.777778	57.388889	5.66	0.0682
Error: MS(temp*conc)	4	40.555556	10.138889		

Source	DF	Type III SS	Mean Square	F Value	Pr > F
temp*conc	4	40.555556	10.138889	1.45	0.2951
Error: MS(Error)	9	63.000000	7.000000		

14.1. The Two-Stage Nested Design.

SAS Code

```
data NESTED;
input make model score @@;
datalines;
1 1 62 1 1 67 1 2 77 1 2 73 2 1 72 2 1 69
2 2 58 2 2 57 3 1 94 3 1 90 3 2 85 3 2 88
;
proc sort;
   by make model;

proc nested;
      class  make model;
        var score;
run;
```

- -

SAS Output

```
                            The NESTED Procedure

                    Coefficients of Expected Mean Squares

                    Source       make       model      Error

                    make            4           2          1
                    model           0           2          1
                    Error           0           0          1


            Nested Random Effects Analysis of Variance for Variable score

  Variance                Sum of                    Error                  Variance
Percent
  Source          DF        Squares  F Value  Pr > F  Term     Mean Square     Component  of
Total

  Total           11    1748.666667                             158.969697    204.104167
100.0000
  make             2    1401.166667     6.79  0.0770  model     700.583333    149.354167
73.1755
  model            3     309.500000    16.29  0.0027  Error     103.166667     48.416667
23.7215
  Error            6      38.000000                               6.333333      6.333333
3.1030


                          score Mean                          74.33333333
                          Standard Error of score Mean         7.64080784
```

15.3. Analysis of Covariance (ANCOVA).
SAS Code

```
data ancova;
input shop x y;
xcentered=x-78.75;
cards;
1 110 40
1 75  38
1 93  30
1 98  47
2 60  25
2 75  32
2 38  13
2 140 35
3 62  27
3 90  24
3 45  20
3 59  13
;

proc glm;
class shop;
model y= shop xcentered;
run;
```

- -

SAS Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	883.871527	294.623842	7.26	0.0114
Error	8	324.795140	40.599392		
Corrected Total	11	1208.666667			

R-Square	Coeff Var	Root MSE	y Mean
0.731278	22.22709	6.371765	28.66667

Source	DF	Type I SS	Mean Square	F Value	Pr > F
shop	2	665.1666667	332.5833333	8.19	0.0116
xcentered	1	218.7048601	218.7048601	5.39	0.0488

Source	DF	Type III SS	Mean Square	F Value	Pr > F
shop	2	288.1556320	144.0778160	3.55	0.0788
xcentered	1	218.7048601	218.7048601	5.39	0.0488

The Cross-Over Design.
SAS Code

```
data reaction_time;
input group subject period drug $ time;
cards;
1 1 1 A 30
1 2 1 A 57
1 3 1 A 52
1 4 1 A 66
1 1 2 B 28
1 2 2 B 50
1 3 2 B 38
1 4 2 B 49
2 1 1 B 34
2 2 1 B 32
2 3 1 B 18
2 4 1 B 28
2 1 2 A 41
2 2 2 A 21
2 3 2 A 27
2 4 2 A 36
;

proc glm data=reaction_time;
class group subject period drug;
model time=group subject(group) period drug/SS3;  /* SS3 = Type III SS
*/
random subject(group);
run;
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
- - -
```

SAS Output

The GLM Procedure

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	9	2539.562500	282.173611	8.24	0.0092
Error	6	205.375000	34.229167		
Corrected Total	15	2744.937500			

R-Square	Coeff Var	Root MSE	time Mean
0.925180	15.42160	5.850570	37.93750

Source	DF	Type III SS	Mean Square	F Value	Pr > F
group	1	1105.562500	1105.562500	32.30	0.0013
subject(group)	6	1212.875000	202.145833	5.91	0.0242
period	1	45.562500	45.562500	1.33	0.2925
drug	1	175.562500	175.562500	5.13	0.0641