STAT 485 Actuarial Science: Financial Mathematics

1.1.1 Effective Rates of Interest

Definition An **interest** is money earned by deposited funds.

Definition An **interest rate** is the rate at which interest is paid to the lender.

Definition A **compound interest** arises when interest is added to the initial investment (called **principal**), so that from that moment on, the interest that has been added itself earns interest.

Example 1.1 The rate of interest is 9% credited annually. The initial deposit is $1,000. Assuming that there are no transactions on the account other than the annual crediting of interest, determine the account balance after interest is credited at the end of three years.

Solution. After one year, the account balance is $1,000(1+0.09)=$1,090. After two years, the account balance is $1,190(1+0.09)=$1,188.10. After three years, the account balance is $1,188.10(1+0.09)=$1,295.03.
Let C denote the initial investment ($C = \$1,000$), and let i be the annual interest rate ($i = 0.09$). Then after n years($n = 3$), the account balance is $C(1+i)(1+i)...(1+i) = C(1+i)^n$. In our case, $C(1+i)^n = \$1,000(1+0.09)^3 = \$1,295.03$. □

In practice interest may be credited monthly or weekly or daily. Each interest rate may be recalculated into an **equivalent** annual rate. For example, a 0.75% monthly interest rate is equivalent to ($(1 + 0.0075)^{12} = 1.0938$) 9.38% annual rate. This annual rate is called an **effective annual rate of interest**.

Definition Two rates of interest are said to be **equivalent** if they result in the same accumulated values at each point in time.

1.1.2 Compound Interest

Example 1.3 A person deposits $1,000 into an account on January 1, 2007. The account credits interest at an effective annual rate of 5% every December 31. The person withdraws $200 on January 1, 2009, deposits $100 on January 1, 2010, and withdraws $250 on January 1, 2012. What is the balance in the account just after interest in credited on December 31, 2013?

Solution. One approach is to recalculate the balance after every transaction.
- On December 31, 2008 the balance is $\$1,000(1 + 0.05)^2 = \$1,102.50$.
- On January 1, 2009 the balance is $\$1,102.50 - \$200 = \$902.50$.

- On December 31, 2009 the balance is $\$902.50(1+0.05) = \$947.63$.
- On January 1, 2010 the balance is $\$947.63 + \$100 = \$1,047.63$.
- On December 31, 2011 the balance is $\$1,047.63(1+0.05)^2 = \$1,155.01$.
- On January 1, 2012 the balance is $\$1,155.01 - \$250 = \$905.01$.
- On December 31, 2013 the balance is $\$905.01(1+0.05)^2 = \$997.77$.

An alternative approach is to accumulate each transaction to the December 31, 2013 date of valuation and then combine all accumulated values, adding deposits and subtracting withdrawals. We have $(\$1,000)(1+0.05)^7 + (\$100)(1+0.05)^4 - (\$200)(1+0.05)^5 - (\$250)(1+0.05)^2 = \$997.77$ □.

Definition The **accumulation factor** from time 0 to time t is the accumulated value at time t of an investment of \$1 made at time 0. The accumulation factor will be denoted by $a(t)$.

Definition The **accumulated amount function**, denoted by $A(t)$, is the accumulated amount of an investment at time t, $A(t) = A(0)a(t)$.

Example Under compound interest, the accumulation factor from 0 to t is $a(t) = (1+i)^t$ where i is the effective annual rate of interest, and time t is a positive real number. □

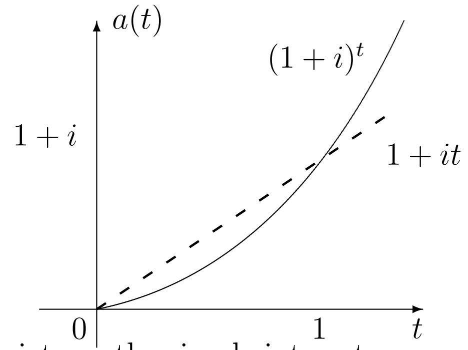
Example Under compound interest, the accumulated amount by time t satisfies the equation $A(t) = A(0)(1+i)^t$. Given any of the three quantities $A(t), A(0), i$, and t, it is possible to find the fourth. The formulas are $A(t) = A(0)(1+i)^t$, $A(0) = A(t)(1+i)^{-t}$, $t = \ln\big(A(t)/A(0)\big)/\ln(1+i)$, and $i = \big(A(t)/A(0)\big)^{1/t} - 1$. For instance, find the time it takes for an initial investment of \$100 to grow to \$150 under the compound annual rate of 5%. Solution: $t = \ln(150/100)/\ln(1.05) = 8.31$. □

In practice, transactions can take place any time during a year. If time is measured in months, it is common to express t in the form $t = m/12$ years. If time is measured in days, use $t = d/365$ years.

1.1.3 Simple Interest

Definition **Simple interest** is an interest that is paid only on the principal amount. The accumulation function from time 0 to time t at annual simple interest rate i is $a(t) = 1 + it$ where t is measured in years.

Comparison of Compound and Simple Interests

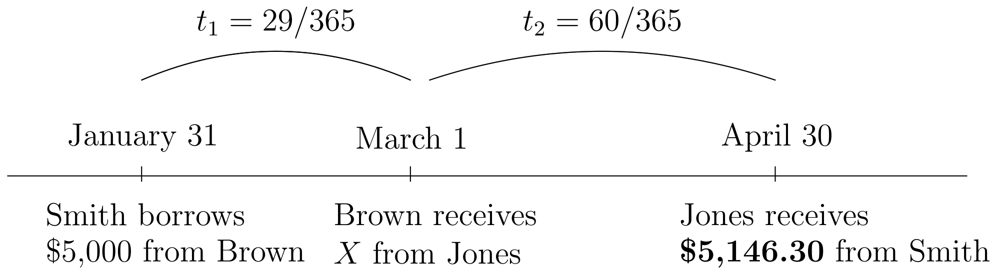


From the picture, the simple interest accumulation factor is larger than that for compound interest if $0 \leq t \leq 1$, and is smaller for $t > 1$.

Interest accumulation is often based on a combination of simple and compound interests. Compound interest would be applied over the completed number of years (that is, interest compounding periods), and simple interest would be applied from then to the fractional point in the current interest period. For example, at annual rate 9% ($i = 0.09$), over a period of 4 years and 5 months, an investment of \$1,000 will grow to $\$1,000(1+i)^4(1+it)$ where $t =$5 months= 5/12 years, that is, it will grow to $\$1,000(1.09)^4\big(1+(5/12)(0.09)\big) = \$1,464.52$ with the accumulation factor of 1.46452.

Definition A **promissory note** is a short-term contract requiring the issuer of the note (the borrower) to pay the holder of the note (the lender) a principal amount plus interest on that principal at a specified annual interest rate for a specified length of time. The length of time is usually less than one year, and the interest is calculated on the basis of simple interest. The interest rate earned by the lender is sometimes referred to as the **yield rate** of the investment.

Example 1.4 On January 31 Smith borrows \$5,000 from Brown and gives Brown a promissory note. The note states that the loan will be repaid on April 30 of the same year, with annual interest at 12%. On March 1 Brown sells the promissory note to Jones, who pays Brown amount of money X in return for the right to collect the payment from Smith on April 30. Jones pays Brown an amount such that Jones' yield from March 1 to the maturity date (April 30) can be stated as an annual rate of interest of 15%. Assume all calculations are based on simple interest and a 365-day year.



(a) Determine the amount Smith was to have paid Brown on April 30.

SOLUTION The payment required at the maturity date is $\$5,000(1+(0.12)(89/365)) = \$5,146.30$.

(b) Determine X, the amount Jones paid to Brown.

SOLUTION The period from March 1 to April 30 is $t_2 = 60/365$. X satisfied the equation $\$5,146.30 = X(1 + (0.15)(60/365))$. Thus, X=\$5,022.46.

(c) Determine the yield rate Brown earned.

SOLUTION Let $t_1 = 29/365$ denote the period from January 31 to March 1, and let j_1 be the annual yield rate earned by Brown. Then j_1 solves the equation $X = \$5,000(1+j_1\,t_1)$ or $j_1 = \big[\big(\$5,022.46/\$5,000\big) - 1\big]/(29/365) = 0.0565$ or 5.65%. □

1.2 Present Value

DEFINITION Let X be the amount that must be invested at the start of a time period to accumulate \$1 at the end of the period at effective annual interest rate i. Then X satisfies the equation $X(i + i) = 1$, or equivalently, $X = 1/(1 + i)$. The quantity $v = 1/(1 + i)$ is called the **present value of an amount of \$1 due in one time period** (or **present value factor** or **discount factor**).

EXAMPLE 1.5 A person invests \$$X$ in a risky fund, which average annual rate of return is 19.5%. What amount must the person invest in order to accumulate \$1,000,000 in 25 years?

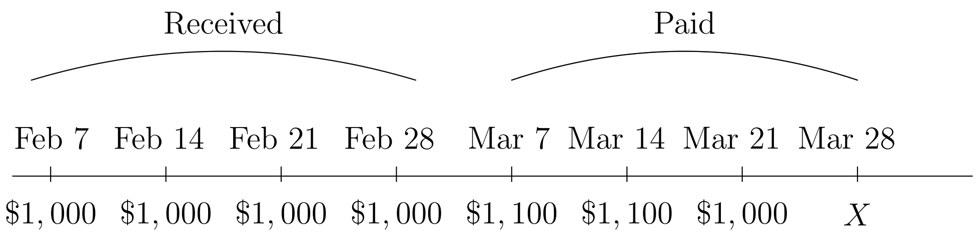
SOLUTION The amount X solves the equation $X(1 + i)^{25} = \$1,000,000$. Thus, $X = \$1,000,000\,v^{25} = \$1,000,000(1.195)^{-25} = \$11,635.96$. □

1.3 Equation of Value

DEFINITION Any financial transaction can be represented in the form of an equation. The components of the equation are the interest rate and the amounts of money disbursed and received. These amounts are termed **dated cash flows (inflow and outflow)**. The equation is called an **equation of value** for the transaction, because it balances the accumulated and present values of the payments made at different time points.

In order to formulate an equation of value for a transaction, it is necessary to choose a reference time point called **valuation date**. At the reference time point, the equation of value balances (1) the accumulated value of all payments already disbursed plus the present value of all payments yet to be disbursed, and (2) the accumulated value of all payments already received plus the present value of all payments yet to be received.

EXAMPLE 1.7 On February 7,14,21 and 28, a person draws \$1,000 credit that charges an effective weekly interest rate of 8% on all credit extended. He pays his debt in four installments which he makes on March 7, 14,21, and 28. He pays \$1,100 on each March 7, 14, and 21. How much must he pay on March 28 to repay his debt completely?



SOLUTION 1 (STRAIGHTFORWARD) Walt's debt on February 7 is \$1,000; on February 14, $\$1,000(1+i)+\$1,000$; on February 21, $\big(\$1,000(1+i)+\$1,000\big)(1+i)+\$1,000 = \$1,000\big((1+i)^2+(1+i)+1\big)$; on February 28, $\$1,000\big((1+i)^2+(1+i)+1\big)(1+i)+\$1,000 = \$1,000\big((1+i)^3+(1+i)^2+(1+i)+1\big)$. On March 7, the debt is $\$,1000\big((1+i)^3+(1+i)^2+(1+i)+1\big)(1+i)-\$1,100 = \$,1000\big((1+i)^4+(1+i)^3+(1+i)^2+(1+i)\big)-\$1,100$; on March 14, $\Big(\$,1000\big((1+i)^4+(1+i)^3+(1+i)^2+(1+i)\big)-\$1,100\Big)(1+i)-\$1,100 = \$,1000\big((1+i)^5+(1+i)^4+(1+i)^3+(1+i)^2\big)-\$1,100\big((1+i)+1\big)$; on March 21, $\Big(\$,1000\big((1+i)^5+(1+i)^4+(1+i)^3+(1+i)^2\big)-\$1,100\big((1+i)+1\big)\Big)(1+i)-\$1,100 = \$,1000\big((1+i)^6+(1+i)^5+(1+i)^4+(1+i)^3\big)-\$1,100\big((1+i)^2+(1+i)+1\big)$; on March 28; $X = \Big(\$,1000\big((1+i)^6+(1+i)^5+(1+i)^4+(1+i)^3\big)-\$1,100\big((1+i)^2+(1+i)+1\big)\Big)(1+i) = \$,1000\big((1+i)^7+(1+i)^6+(1+i)^5+(1+i)^4\big)-\$1,100\big((1+i)^3+(1+i)^2+(1+i)\big) = \$2,273.79$.

SOLUTION 2 Choose March 28 as the reference time point for valuation. Then the accumulated value of all payments received on February 7, 14, 21, 28 are $\$1,000(1+i)^7$, $\$1,000(1+i)^6$, $\$1,000(1+i)^5$, and $\$1,000(1+i)^4$, respectively. The accumulated value of all payments paid on March 7, 14, 21, and 28 are, respectively, $\$1,100(1+i)^3$, $\$1,100(1+i)^2$, $\$1,100(1+i)$, and X. The equation of value is the difference between the accumulated values of all the sums received and all the sums paid, that is, the equation of value is

$$\$1,000\Big[(1+i)^7+(1+i)^6+(1+i)^5+(1+i)^4\Big]-\$1,100\Big[(1+i)^3+(1+i)^2+(1+i)\Big]-X = 0\,.$$

From here $X = \$2,273.79$.

SOLUTION 3 Choose February 7 as the reference time point. Then the present value of the sums that will be received on February 7, 14, 21, 28 are $\$1,000$, $\$1,000/(1+i) = \$1,000v$, $\$1,000/(1+i)^2 = \$1,000v^2$, and $\$1,000/(1+i)^3 = \$1,000v^3$, respectively, where v is the present value factor. The present value of the sums that will be paid on March 7, 14, 21, and 28 are, respectively, $\$1,100/(1+i)^4 = \$1,100v^4$, $\$1,100/(1+i)^5 = \$1,100v^5$, $\$1,100/(1+i)^6 = \$1,100v^6$, and Xv^7. The equation of value is the difference between the present values of all the sums that will be received and all the sums that will be paid, that is, the equation of value is

$$\$1,000\big[1+v+v^2+v^3\big] - \$1,100\big[v^4+v^5+v^6\big] - Xv^7 = 0\,.$$

From here $X = \$2,273.79$.

SOLUTION 4 Choose February 21 as the reference time point. Then the accumulated value of the payments received on February 7 and 14 are $\$1,000(1+i)^2$ and $\$1,000(1+i)$, respectively. The present value of the payments that will be received on February 21 and 28 are $\$1,000$ and $\$1,000/(1+i)$, respectively. The present value of the amounts that will be paid on March 7, 14, 21, and 28 are, respectively, $\$1,100/(1+i)^2$, $\$1,100/(1+i)^3$, $\$1,100/(1+i)^4$, and $X/(1+i)^5$. The equation of value is

$$\$1,000\Big[(1+i)^2+(1+i)+1+\frac{1}{(1+i)}\Big]-\$1,100\Big[\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\frac{1}{(1+i)^4}\Big]-\frac{X}{(1+i)^5}=0\,.$$

From here $X = \$2,273.79$. □

1.4 NOMINAL RATES OF INTEREST

DEFINITION A **nominal annual interest rate** $i^{(m)}$ **compounded** m **times per year** refers to an interest rate $i^{(m)}/m$ compounded every $1/m$th of a year.

EXAMPLE 1.8 A credit card account charges 24% on unpaid balances, payable monthly. The quoted rate of 24% is a nominal annual interest rate. A person puts \$1,000 on his credit card, and doesn't pay for one year.
(a) How much does he owe according to his 13th statement?
(b) Compute an effective annual interest rate.

SOLUTION (a) The monthly interest rate is $i^{(12)} = 24/12 = 2\%$. His 1st statement is for \$1,000 and doesn't charge interest. His 13th statement is for $\$1,000(1.02)^{12} = \$1,268.23$.

(b) The effective annual rate of interest is 26.823%. □

DEFINITION A nominal annual interest rate is sometimes called an **annual percentage rate (APR)**. The corresponding monthly interest rate is called a **periodic rate**.

EXAMPLE 1.9 Bank A offers an annual interest rate of 15.25% compounded semiannually, and Bank B offers an annual rate of 15% compounded monthly. Which bank offers a higher effective annual interest rate?

SOLUTION The periodic rate in Bank A is $i_A^{(2)} = 15.25\%/2 = 7.625\%$. Thus, a deposit of \$1 in Bank A will grow to $(1 + 0.07625)^2 = 1.158314$. The periodic rate in Bank B is $i_B^{(12)} = 15\%/12 = 1.25\%$. A deposit of \$1 will grow to $(1.0125)^{12} = 1.160755$. Hence, the effective annual interest rates in Bank A and B are 15.8314% and 16.0755%, respectively. Bank B has a higher effective annual interest rate. □

The formula that connects the nominal annual interest rate $i^{(m)}$ and the effective annual interest rate i is

$$\big(1 + \frac{i^{(m)}}{m}\big)^m = 1 + i\,.$$

EXAMPLE 1.10 Suppose an effective annual rate of interest is 12%. Find the equivalent nominal annual rate for $m = 1, 12, 365$, and ∞.

SOLUTION Using the formula $i^{(m)} = m\big[(1+i)^{1/m} - 1\big]$, we compute $i^{(1)} = i = 0.12$, $i^{(12)} = 0.1139$, $i^{(365)} = 0.113346$. When $m \to \infty$, $m\big[(1+i)^{1/m} - 1\big] \to \ln(1+i) = 0.113329$. The limiting case $m \to \infty$ is called **continuous compounding**. □

1.5 Effective and Nominal Rates of Discount

DEFINITION An interest payable at the end of an interest period is called **interest payable in arrears**.

DEFINITION An interest payable at the beginning of an interest period is called **interest payable in advance**.

DEFINITION The rate of interest payable in advance is called the **rate of discount**.

EXAMPLE. A person borrows \$1,000 for one year at an interest rate of 10% payable in advance. It means that he receives the loan amount of \$1,000 but must immediately pay the lender \$100. Thus, the net loan amount is \$900. One year later he must repay \$1,000. The effective annual rate of interest of this transaction is 100/900=0.1111 or 11.11%. □

DEFINITION The **effective annual rate of discount** from time 0 to time 1 is

$$d = \frac{A(1) - A(0)}{A(1)}$$

where $A(t)$ is the accumulated amount function.

EXAMPLE In our example, $A(1) = 1,000$, $A(0) = 900$, $d = (1000{-}900)/1000 = 100/1000 = 0.10$ or 10%. □

Recall that the effective annual rate of interest satisfies the equality $A(1) = A(0)(1 + i)$. Therefore,

$$i = \frac{A(1) - A(0)}{A(0)}\,.$$

From here, the equalities hold

$$d = \frac{i}{1+i} \text{ and } i = \frac{d}{1-d}\,.$$

EXAMPLE In our example, $i = 0.1/(1 - 0.1) = 1/9 = 0.1111$, and $d = (1/9)/(1 + 1/9) = 1/10 = 0.10$. □

DEFINITION A **nominal annual discount rate** $d^{(m)}$ **compounded** m **times per year** refers to a discount rate $d^{(m)}/m$ compounded every $1/m$th of a year.

The relation between d and $d^{(m)}$ is

$$1 - d = \Big(1 - \frac{d^{(m)}}{m}\Big)^m\,.$$

PROOF: By definition, $A(0) = A(1)(1 - d)$. Hence,

$$A(0) = A(1/m)(1{-}d^{(m)}/m) = A(2/m)(1{-}d^{(m)}/m)^2 = ... = A(1)(1{-}d^{(m)}/m)^m,$$

which proves the formula.

EXAMPLE 1.12 Suppose an effective annual rate of interest is 12%. Find the equivalent nominal annual rate of discount for $m = 1, 12, 365$, and ∞.

SOLUTION First compute the effective annual rate of discount

$$d = \frac{i}{1+i} = \frac{0.12}{1.12} = 0.107143\,.$$

Using the formula $d^{(m)} = m\big[1 - (1 - d)^{1/m}\big]$, we compute $d^{(1)} = d = 0.107143$, $d^{(12)} = 0.1128$, $d^{(365)} = 0.1133$. When $m \to \infty$, $m\big[1 - (1 - d)^{1/m}\big] \to -\ln(1 - d) = 0.113329 = \ln(1 + i)$. □

Show that is it always so: $-\ln(1 - d) = \ln(1 + i)$.

1.6 The Force of Interest

The interest rate by the investment for the $1/m$th of a year period from time t to time $t + 1/m$ is

$$\frac{A(t + 1/m) - A(t)}{A(t)}.$$

The nominal annual rate of interest compounded m times per year is

$$i^{(m)} = m\,\frac{A(t + 1/m) - A(t)}{A(t)}.$$

Letting $m \to \infty$, we get the nominal annual interest rate **compounded continuously**,

$$i^{(\infty)} = \lim_{m\to\infty} m\,\frac{A(t + 1/m) - A(t)}{A(t)} = \frac{A'(t)}{A(t)}.$$

EXAMPLE 1.13 From Example 1.10, $i^{(\infty)} = \ln(1 + i)$. Indeed, $A(t) = A(0)(1 + i)^t$, thus, $A'(t) = A(0)(1 + i)^t \ln(1 + i) = A(t)\ln(1 + i)$. Hence, $i^{(\infty)} = A'(t)/A(t) = \ln(1 + i)$. □

DEFINITION The **force of interest** at time t is

$$\delta_t = \frac{A'(t)}{A(t)}.$$

To express the accumulated amount function $A(t)$ in terms of the force of interest, solve the differential equation

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt}\ln[A(t)].$$

The solution is $A(t) = A(0)\exp\left\{\int_0^t \delta_u\,du\right\}$.

EXAMPLE 1.14 Given $\delta_t = 0.08 + 0.005t$, find an accumulated value over five years of an investment of \$1,000 made at time (a) t=0, (b) t=2.

SOLUTION (a) $A(0) = \$1,000$, and $A(5) = A(0)\exp\left\{\int_0^5 \delta_u\,du\right\} = 1000\exp\left\{\int_0^5 (0.08 + 0.005u)du\right\} = 1000\exp\{0.4625\} = 1588.04$.

(b) $A(2) = 1000$, and $A(7) = A(2)\exp\left\{\int_2^7 (0.08 + 0.005u)du\right\} = 1669.46$. □

If the force of interest is constant, $\delta_t = \delta$, then $A(t) = A(0)\exp\left\{\int_0^t \delta_u\,du\right\} = \exp\{\delta\,t\}$. On the other hand, we know that $A(t) = A(0)(1+i)^t$ for a constant effective annual interest rate i. Hence, $\delta = \ln(1 + i)$ or $i = e^\delta - 1$.

1.7 Inflation and the "Real" Rate of Interest

Definition **Inflation** is a rise in the general level of prices of goods and services in an economy over a period of time.

Definition A chief measure of price inflation is the **inflation rate**, the percentage change in the Consumer Price Index (CPI) over time, generally quoted on an annual basis. The change in the CPI measures the effective annual rate of change in the cost of a specified **basket** of consumer goods and services.

Definition (Explanation is in Example 1.16). With effective annual interest rate i and annual inflation rate r, the real rate of interest for the year is

$$i_{\text{real}} = \frac{\text{value of real return in the end of the year dollars}}{\text{value of invested amount in theend of the year dollars}} = \frac{i-r}{1+r}.$$

Example 1.16 A person invests \$1,000 for one year at effective annual interest rate of 15.5%. At the time Smith makes the investment, the cost of a certain consumer item is \$1. One year later, the interest is paid and the principal returned. Assume that the inflation is 10%.

(a) What is the annual growth rate in **purchasing power** (the number of items that can purchased) with respect to the consumer item?

Solution (a) At the start of the year, Smith can buy 1,000 items. At the end of the year, he receives $\$1,000(1.155) = \$1,155$. Due to inflation, the cost of the item at the end of the year is \$1.10. Smith is able to buy \$1,155.00/\$1.10=1,050 items. Thus, his purchasing power has grown by (1050-1000)/1000x100%=5%.

(b) Compute the real rate of interest.

Solution $i_{\text{real}} = (0.155 - 0.1)/(1 + 0.1) = 0.055/1.1 = 0.05$ or 5%.

Note that the real interest rate can be defined as the rate of change in purchasing power. Indeed, the general formula for the rate of change in purchasing power is

$$\frac{\frac{A(0)(1+i)}{P(1+r)} - \frac{A(0)}{P}}{\frac{A(0)}{P}} = \frac{1+i}{1+r} - 1 = \frac{i-r}{1+r}. \quad \square$$

2.1 Level Payment Annuities

Definition An **annuity** is a series of periodic payments.

Definition An **annuity-certain** is a series of payments that are definitely made. They are not contingent upon occurrence of any random event.

Definition An **annuity-immediate** is a series of payments made at the end of each period.

Definition An **annuity-due** is a series of payments made at the beginning of each period.

For now we will study annuity-immediate.

Example 2.1 A person deposits \$30 in a bank account on the last day of each month. The annual interest rate is 9% compounded monthly, and the interest is paid into the account on the last day of each month. Find the account balance after the 140th deposit.

Solution Let $j = i^{(12)} = 0.09/12 = 0.0075$ or 0.75%. The account balance after the 1st deposit is \$30; after the 2nd one, $\$30(1+j) + \$30 = \$30\big[(1+j)+1\big]$; after the 3rd, $\$30\big[(1+j)+1\big](1+j)+\$30 = \$30\big[(1+j)^2+(1+j)+1\big]$. After the 140th deposit, the balance is

$$\$30\Big[(1+j)^{139}+(1+j)^{138}+...+(1+j)+1\Big] = \$30\left[\frac{(1+j)^{140}-1}{(1+j)-1}\right] = \$7,385.91. \quad \square$$

Definition The **accumulated value of an n-payment annuity-immediate of \$1 per period** is

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

where i is the constant interest rate per payment period.

Example In our example, $s_{\overline{140}|0.0075} = \$7,385.91/30 = \$246.197$. $\square$

Example After the n payments are completed, the balance $s_{\overline{n}|i}$, if not withdrawn, will continue to accumulate with interest only. After k time periods, the balance will be

$$s_{\overline{n}|i}(1+i)^k = \frac{(1+i)^n - 1}{i}(1+i)^k = \frac{(1+i)^{n+k} - (1+i)^k}{i}$$

$$= \frac{(1+i)^{n+k} - 1}{i} - \frac{(1+i)^k - 1}{i} = s_{\overline{n+k}|i} - s_{\overline{k}|i}\,.$$

Useful formula:

$$s_{\overline{n+k}|i} = s_{\overline{n}|i}(1+i)^k + s_{\overline{k}|i}\,.$$

Note that n and k are switched in this formula in the textbook. $\square$

Extensions of this formula prove to be useful.

Example 2.4 (*Annuity accumulation with non-level interest rate*) Note that "non-level" means "non-constant". Suppose in Example 2.1, after 68 months, the nominal interest rate changes to 7.5% still compounded monthly. Find the balance after 140 months.

Solution After 68 months, the balance is $\$30\, s_{\overline{68}|0.0075} = \$2,648.50$. During the next 72 months, the balance accumulates to $\$2,648.50(1+0.75/12)^{72} = \$2,648.50(1.00625)^{72} = \$4,147.86$.
The remaining deposits continuing for the last 72 months accumulate to $\$30\, s_{\overline{72}|0.00625} = \$2,717.36$. Thus, the total balance is $\$30\Big[s_{\overline{68}|0.0075}(1.00625)^{72} + s_{\overline{72}|0.00625}\Big] = \$6,865.22$. □

Example 2.5 (*Annuity accumulation with a changing payment*) Suppose that 10 monthly payments of \$50 each are followed by 14 monthly payments of \$75 each. If the effective monthly rate is 1%, what is the accumulated value the annuity at the time of the final payment?

Solution At the time of the final payment, 24 months later, the accumulated value of the first 10 payments is $\$50\, s_{\overline{10}|0.01}\,(1.01)^{14} = \$601.30$. The value of the final 14 payments, also valued at time 24 months is $\$75\, s_{\overline{14}|0.01} = \$1,121.06$. Thus, the total balance is \$1,722.36. □

2.1.2 Present Value of an Annuity

Example 2.6 The 1st withdrawal from an account will take place one year from now. There will be four yearly withdrawals of \$1,000 each. The account has an effective annual interest rate of 6%. Calculate an amount of a single deposit today so that the account balance will be reduced to 0 after the 4th withdrawal.

Solution Suppose that the amount of the initial deposit is $\$X$. The account balance after the 1st withdrawal is $\$X(1.06) - \$1,000$. The balance after the 2nd withdrawal is $\big[\$X(1.06) - \$1,000\big](1.06) - \$1,000 = \$X(1.06)^2 - \$1,000(1.06) - \$1,000$. The balance after the 3rd withdrawal is (note a typo in the text) $\big[\$X(1.06)^2 - \$1,000(1.06) - \$1,000\big](1.06) - \$1,000 = \$X(1.06)^3 - \$1,000(1.06)^2 - \$1,000(1.06) - \$1,000$. The balance after the 4th withdrawal is $\big[\$X(1.06)^3 - \$1,000(1.06)^2 - \$1,000(1.06) - \$1,000\big](1.06) - \$1,000 = \$X(1.06)^4 - \$1,000(1.06)^3 - \$1,000(1.06)^2 - \$1,000(1.06) - \$1,000$. Since the final balance is zero, X solves

$$\$X(1.06)^4 = \$1,000(1.06)^3 + \$1,000(1.06)^2 + \$1,000(1.06) + \$1,000,$$

or equivalently,

$$\$X = \frac{\$1,000}{1.06} + \frac{\$1,000}{(1.06)^2} + \frac{\$1,000}{(1.06)^3} + \frac{\$1,000}{(1.06)^4} = \$1,000\big[v + v^2 + v^3 + v^4\big]$$

$$= \$1,000\, v\left[1 + v + v^2 + v^3\right] = \$1,000\, v\,\frac{1 - v^4}{1 - v} = \$1,000\frac{1}{1+i}\,\frac{1 - v^4}{1 - \frac{1}{1+i}}$$

$$= \$1,000\frac{1 - v^4}{i} = \$1,000\,\frac{1 - (1.06)^{-4}}{0.06} = \$3,465.11\,. \ \square$$

Definition The **present value of an n-payment annuity-immediate of \$1 per period** is

$$a_{\overline{n}|i} = v + v^2 + \cdots + v^n = \frac{1 - v^n}{i}$$

where i is the constant interest rate per payment period.

Example 2.7 (*Loan repayment*) A person takes a \$12,000 loan and has to make monthly payments for 3 years, starting one month from now, with a nominal interest rate of 12% compounded monthly. Find a monthly payment on this loan and the total amount of payment.

Solution Let $\$P$ be a monthly payment. It solves $\$12,000 = \$P\, a_{\overline{36}|0.01}$. Hence,

$$\$P = \frac{\$12,000}{a_{\overline{36}|0.01}} = \frac{\$12,000}{(1 - (1.01)^{-36})/0.01} = \$398.57$$

and the total payment is $\$398.57(36) = \$14,348.52$. $\square$

Definition A **k-period deferred, n-payment annuity of \$1 per period** is a series of n payments that start $k+1$ time periods from now.

Example 2.8 Suppose in the previous example, the 1st payment starts 9 months from now. Find monthly and total payments.

Solution A monthly payment $\$P$ satisfies

$$\$12,000 = \$P\left[v^9 + v^{10} + \cdots + v^{44}\right] = \$P\, v^8\, a_{\overline{36}|0.01}\,.$$

Thus,

$$\$P = \frac{\$12,000}{v^8\, a_{\overline{36}|0.01}} = (1.01)^8\,\frac{\$12,000}{a_{\overline{36}|0.01}} = (1.01)^8(\$398.57) = \$431.60,$$

and the total payment is $\$431.60(36) = \$15,537.60$. $\square$.

From the relation $v^k\left[v + v^2 + \cdots + v^n\right] = v^k\, a_{\overline{n}|i}$, we get

$$v^k\, a_{\overline{n}|i} = v^k\,\frac{1 - v^n}{i} = \frac{v^k - v^{n+k}}{i} = \frac{1 - v^{n+k}}{i} - \frac{1 - v^k}{i} = a_{\overline{n+k}|i} - a_{\overline{k}|i}\,.$$

A useful formula is $a_{\overline{n+k}|i} = v^k\, a_{\overline{n}|i} + a_{\overline{k}|i}$.

EXAMPLE A person takes a \$12,000 loan and has to make monthly payments for the first year at the interest rate of 10% compounded monthly, and for the following 2 years with the interest rate of 12% compounded monthly. The first payment is due one month from now. Find a monthly payment on this loan and the total amount of payment.

SOLUTION Let $\$P$ be a monthly payment. The effective monthly interest rate for the 1st year is $i_1 = 0.10/12 = 0.0083$, while for the next two years, it is $i_2 = 0.12/12 = 0.01$. The amount of monthly payment $\$P$ solves the equation

$$\$12,000 = \$P\left[a_{\overline{12}|i_1} + (1+i_1)^{-12} a_{\overline{24}|i_2}\right]$$

$$= \$P\left[\frac{1-(1+i_1)^{-12}}{i_1} + (1+i_1)^{-12}\frac{1-(1+i_2)^{-24}}{i_2}\right] = (30.6043)\$P,$$

Thus, $\$P = \$392.10$ and the total amount paid is $(\$392.10)(36) = \$14,115.60$. □

2.1.2.3. RELATIONSHIP BETWEEN $a_{\overline{n}|i}$ AND $s_{\overline{n}|i}$.

Recall that $s_{\overline{n}|i}$ is the accumulated value of an annuity-immediate at the time of the last nth payment. The quantity $a_{\overline{n}|i}$ is the present value of an n-payment annuity-immediate one period before the 1st payment. Thus, the valuation point of the present value is n periods earlier than that for the accumulated value. Therefore,

$$s_{\overline{n}|i} = (1+i)^n \, a_{\overline{n}|i} \, ,$$

or equivalently,

$$a_{\overline{n}|i} = v^n \, s_{\overline{n}|i} \, .$$

This equality can be verified algebraically by observing that

$$v^n \, s_{\overline{n}|i} = \frac{1}{(1+i)^n}\left[\frac{(1+i)^n - 1}{i}\right] = \frac{1-v^n}{i} = a_{\overline{n}|i} \, .$$

2.1.2.4 VALUATION OF PERPETUITIES

Note that

$$\lim_{n\to\infty} a_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i} \, .$$

This expression can also be derived by summing the infinite series of present values of payments

$$v + v^2 + v^3 + + \cdots = v(1 + v + v^2 + \cdots) = v\frac{1}{1-v} = \frac{1}{1+i}\frac{1}{1-\frac{1}{1+i}} = \frac{1}{i} \, .$$

DEFINITION An annuity that has no definite end is called a **perpetuity**.

NOTATION The present value of a perpetuity is denoted by $a_{\overline{\infty}|i} = 1/i$.

Example 2.10 A person deposits \$10,000 in a bank account that offers an effective annual interest rate of 8% (e.g., buys a certified deposit). In one year, he withdraws \$800 of interest. The principal amount of \$10,000 remains in the account and generates \$800 the following year. If the person keeps withdrawing only the interest, the process will go forever. This is what is called "lives off the interest". □

There is another way to derive the expression for $a_{\overline{\infty}|i}$. Let $\$X$ denote the amount that has to be invested to generate \$1 interest per time period. Then $\$X$ solves $\$Xi = \$1$, or $X = 1/i$.

2.1.3 Annuity-Immediate and Annuity-Due

Recall in case of an annuity-immediate payments are made at the end of valuation periods, while an annuity-due payment is made at the beginning of each period.

Definition The present value of an n-payment annuity-due at the time of the 1st payment is

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \cdots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

where $d = i/(1+i) = 1 - 1/(1+i) = 1 - v$ is the rate of discount. The accumulated value one period after the final payment is

$$\ddot{s}_{\overline{n}|i} = (1+i) + (1+i)^2 + \cdots + (1+i)^n = (1+i)\left[\frac{(1+i)^n - 1}{i}\right] = \frac{(1+i)^n - 1}{d}\,.$$

Note the relationships

$$\ddot{a}_{\overline{n}|i} = (1+i)\,a_{\overline{n}|i} \ \text{ and } \ \ddot{s}_{\overline{n}|i} = (1+i)\,s_{\overline{n}|i}\,.$$

Example 2.11 A person makes monthly deposits of \$200 into a fund earning 6% annual interest compounded monthly. The first deposit occurred on January 1, 1995. Compute the accumulated balance on December 31, 2009.

Solution There are 180 deposits. The monthly interest rate is 0.06/12=0.005 or 0.5%. The accumulated value is

$$\$200\,\ddot{s}_{\overline{180}|0.005} = \$200(1.005)\frac{(1.005)^{180}-1}{0.005} = \$58,454.56. \ \square$$

3.1 Amortization Method of Loan Payment

Definition A **loan repayment** involves paying down the principal amount of the loan as well as the accumulating interest on the loan.

DEFINITION An **amortizing loan** is a loan where the principal is paid down over the life of the loan, according to some **amortization schedule**.

The origin of the word " amortize" is Latin *mort* = death.

EXAMPLE Consider a loan of amount \$1,000 with an interest rate of 10% per year. Suppose there is a payment of \$200 at the end of the 1st year, a payment of \$500 at the end of the 2nd year, and a final payment of \$X at the end of the 3rd year. At the end of the 1st year, the amount owed, including interest, is \$1000(1.1)=\$1,100. After a payment of \$200, the outstanding balance is \$900. At the end of the 2nd year, the amount owed is \$900(1.1)=\$990. The payment of \$500 reduces it to \$490. At the end of the 3rd year, the amount owed is \$490(1.1)=\$539. Thus, the amount needed to repay the loan completely is \$539. □

The amortization method requires that whenever a payment is made, interest is paid first, and any amount remaining is applied toward reducing the principal (goes towards principal). In particular, a periodic loan payment cannot be less than accumulated interest for the time period.

EXAMPLE Consider our example from the point of view of interest and principal. The **amortization** of the loan can be summarized in an **amortization schedule**.

Time (in years)	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	\$1,000
1	\$200	\$1,000(0.1)=\$100	\$200-\$100=\$100	\$1000-\$100=\$900
2	\$500	\$900(0.1)=\$90	\$500=\$90=\$410	\$900-\$410=\$490
3	\$539	\$490(0.1)=\$49	\$539-\$49=\$490	\$490-\$490=\$0

Note that the outstanding balance can be updated from one year to the next according to the equalities: $\$1,000(1.1) - \$200 = \$900$, $\$900(1.1) - \$500 = \$490$, $\$490(1.1) - \$539 = \$0$.

Now multiply the first equality by $(1.1)^2$ and the second one by (1.1) and moving all terms to the left-hand side, we get $\$1,000(1.1)^3 - \$200(1.1)^2 - \$900(1.1)^2 = \$0$, $\$900(1.1)^2 - \$500(1.1) - \$490(1.1) = \$0$, $\$490(1.1) - \$539 = \$0$.
Now adding the equalities, we obtain

$$\$1,000(1.1)^3 - \$200(1.1)^2 - \$500(1.1) - \$539 = \$0,$$

or equivalently, $\$1,000 = \$200v + \$500v^2 + \$539v^3$. Hence, the **original loan amount is equal to the present value of the loan payments** (v is the present value factor or discount factor).

Note also that the total amount paid is 200 + \$500 + \$539 = \$1239, which breaks into the original loan amount of \$1,000 and the interest of \$239. Indeed, as seen from the table, the interest paid is \$100+\$90+\$49=\$239. □

Definition An **amortized loan** of amount L made at time 0 with the discount factor v to be repaid in n payments of amounts $K_1,\ K_2, \ldots, K_n$ is based on the equation

$$L = K_1\, v + K_2\, v^2 + \cdots + K_n\, v^n\,.$$

Notation At the time t, K_t is the payment amount, OB_t is the outstanding balance after the payment is made, I_t is the interest on outstanding balance since the previous payment was made, PR_t is the part of the payment K_t that is applied toward repaying loan principal. The amortization schedule in general form is

Time (in years)	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	$L = OB_0$
1	K_1	$I_1 = OB_0 \times i$	$PR_1 = K_1 - I_1$	$OB_1 = OB_0 - PR_1$
		$\cdots$	$\cdots$	
n	K_n	$I_n = OB_{n-1} \times i$	$PR_n = K_n - I_n$	$OB_n = OB_{n-1} - PR_n$

Example 3.1 A loan of amount \$1,000 at a nominal annual interest rate of 12% compounded monthly is repaid by 6 monthly payments, starting one month after the loan is made. The first three payments are \$X each and the final three payments are \$2X each. Construct the amortization schedule for this loan.

Solution The monthly interest rate is $i = 0.12/12 = 0.01$ or 1%. The discount factor is $v = 1/(1+i) = 1/(1.01)$. The amount X solves the equation $\$1,000 = \$X(v + v^2 + v^3) + \$2X(v^4 + v^5 + v^6) = \$X a_{\overline{3}|0.01} + \$2Xv^3 a_{\overline{3}|0.01}$. From here $\$X = \$115.61$ and so $\$2X = \$231.21$ to the nearest cent. The amortization schedule is as follows:

Time (in years)	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	$L = OB_0$ $= \$1,000$
1	K_1 $= \$115.61$	$I_1 = OB_0 \times i$ $= \$10$	$PR_1 = K_1 - I_1$ $= \$105.61$	$OB_1 = OB_0 - PR_1$ $= \$894.39$
2	K_2 $= \$115.61$	$I_2 = OB_1 \times i$ $= \$8.94$	$PR_2 = K_2 - I_2 =$ $\$106.67$	$OB_2 = OB_1 - PR_2 =$ $\$787.72$
3	K_3 $= \$115.61$	$I_3 = OB_2 \times i$ $= \$7.88$	$PR_3 = K_3 - I_3$ $= \$107.73$	$OB_3 = OB_2 - PR_3$ $= \$679.99$
4	K_4 $= \$231.21$	$I_4 = OB_3 \times i$ $= \$6.80$	$PR_4 = K_4 - I_4$ $= \$224.41$	$OB_4 = OB_3 - PR_4$ $= \$455.58$
5	K_5 $= \$231.21$	$I_5 = OB_4 \times i$ $= \$4.56$	$PR_5 = K_5 - I_5$ $= \$226.65$	$OB_4 = OB_4 - PR_5$ $= \$228.93$
6	K_6 $= \$231.21$	$I_6 = OB_5 \times i$ $= \$2.29$	$PR_6 = K_6 - I_6 =$ $\$228.92$	$OB_4 = OB_5 - PR_6$ $= \$0$

The total amount paid on the loan is \$1,040.46, of which \$40.47 is the interest payment and \$1,000 is the principal payment.

3.2 Amortization of a Loan with Level Payments

If a loan is repaid with level payments, the amortization schedule has a systematic form. Suppose the payment is of amount 1, that is, $K_1 = K_2 = \cdots = K_= 1$. The loan amount is the present value of the payments, $L = OB_0 = a_{\overline{n}|i}$. The outstanding balance just after the tth payment is the original loan amount valued at time t minus the accumulated value of the first t payments $OB_t = L(1+i)^t - s_{\overline{t}|i} = a_{\overline{n}|i}(1+i)^t - s_{\overline{t}|i}$. This is the retrospective form of the outstanding balance. To obtain a prospective form, recall that $a_{\overline{n}|i} = (1-(1+i)^{-n})/i$ and $s_{\overline{t}|i} = ((1+i)^t - 1)/i$. Therefore,

$$OB_t = a_{\overline{n}|i}(1+i)^t - s_{\overline{t}|i} = \frac{1-(1+i)^{-n}}{i}\,(1+i)^t \; - \; \frac{(1+i)^t - 1}{i}$$

$$= \frac{(1+i)^t - (1+i)^{-(n-t)}}{i} - \frac{(1+i)^t - 1}{i} = \frac{1-(1+i)^{-(n-t)}}{i} = a_{\overline{n-t}|i}\,,$$

which is the present value at time t of the remaining $n-t$ payments.

The interest paid at time t is $I_t = OB_{t-1}\,i = a_{\overline{n-t+1}|i}\,i = 1 - v^{n-t+1}$. The principal repaid at time t is $PR_t = K_t - I_t = 1 - (1 - v^{n-t+1}) = v^{n-t+1}$. The outstanding balance at time t is

$$OB_t = OB_{t-1} - PR_t = a_{\overline{n-t+1}|i} - v^{n-t+1}$$

$$= \frac{1-(1+i)^{-(n-t+1)}}{i} - (1+i)^{-(n-t+1)} = \frac{1-(1+i)(1+i)^{-(n-t+1)}}{i}$$

$$= \frac{1-(1+i)^{-(n-t)}}{i} = a_{\overline{n-t}|i}\,.$$

The amortization schedule is as follows.

Time (in years)	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	$L = OB_0 = a_{\overline{n}|i}$
1	$K_1 = 1$	$I_1 = OB_0\,i$ $= ia_{\overline{n}|i} = 1 - v^n$	$PR_1 = K_1 - I_1$ $= v^n$	$OB_1 = OB_0 - PR_1$ $= a_{\overline{n}|i} - v^n = a_{\overline{n-1}|i}$
2	$K_2 = 1$	$I_2 = OB_1\,i$ $= ia_{\overline{n-1}|i} = 1 - v^{n-1}$	$PR_2 = K_2 - I_2$ $= v^{n-1}$	$OB_2 = OB_1 - PR_2$ $= a_{\overline{n-1}|i} - v^{n-1} = a_{\overline{n-2}|i}$
		$\cdots$	$\cdots$	
n	$K_n = 1$	$I_n = OB_{n-1}\,i$ $= ia_{\overline{1}|i} = 1 - v$	$PR_n = K_n - I_n$ $= v$	$OB_n = OB_{n-1} - PR_n$ $= a_{\overline{1}|i} - v = 0$

The total interest paid is $I_{total} = (1-v^n)+(1-v^{n-1})+\cdots+(1-v) = n-a_{\overline{n}|i}$. The total amount paid is $K_{total} = K_1+K_2+\cdots+K_n = n$. The total principal repaid is $K_{total} - I_{total} = a_{\overline{n}|i} = L$.

EXAMPLE 3.5 A home buyer borrows \$250,000 for 30-year period with level monthly payments beginning one month after the loan is made. The interest rate is 9% compounded monthly. Find the amortization schedule.

SOLUTION The monthly interest rate is 0.09/12=0.0075 or 0.75%. The monthly payment is K that satisfies $\$Ka_{\overline{360}|0.0075} = \$250,000$. Hence, $K = \$250,000/\big[(1-(1.0075)^{-360})/0.0075\big] = \$2,011.56$. The amortization schedule is (see page 189, show in Excel)

Time (in years)	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	\$250,000
1	$\$2,011.56$	$(\$250,000)(0.0075)$ \$1,875.00	\$2,011.56-\$1,875.00 =\$136.56	\$250,000-\$2,011.56 \$249,863.44
		$\cdots$	$\cdots$	
360	\$2,011.56	\$14.97	\$1,996.58	0

□

3.3 The Sinking-fund Method of Loan Repayment

Consider the case of a loan which calls for periodic payments of interest only during the term of the loan, and a repayment of the principal amount at the end of the term. For such a loan, the borrower has to make n payments of interest of amount $L\,i$, along with a payment of L at time n. The borrower may wish to accumulate the amount L during the term of the loan by means of n periodic deposits into an interest-bearing savings account called a **sinking fund**. The method of loan repayment is called the **sinking-fund**

method. In practice it is usually the case that the interest rate charged by the lender of the loan, i, is higher than that of the sinking fund, j, that is, $i > j$. The borrower's periodic payment to the lender is $L\,i$ and to the sinking fund, $L/s_{\overline{n}|j}$. Thus, the **total periodic outlay** is $L(i + 1/s_{\overline{n}|j})$.

EXAMPLE 3.6 A loan of $100,000 is to be repaid by ten annual payments beginning one year from now. The lender sets the annual rate of 10% and repayment of principal after 10 years. The borrower makes 10 annual deposits to a sinking fund earning 8%.
(a) Find the borrower's total annual outlay.
SOLUTION The total annual outlay is $L(i+1/s_{\overline{n}|j}) = \$100,000(0.1+1/s_{\overline{10}|0.08}) = \$100,000\big(0.1 + 0.08/((1.08)^{10} - 1)\big) = \$16,902.95$.

(b) Find the level annual payment required by the amortization method at 10%.
SOLUTION $K = \$100,000/a_{\overline{10}|0.1} = \$16,274.54$. Note that the payment is lower than that for the sinking-fund method.

(c) Find the rate of interest i' for which the amortization method results in the total annual outlay of $16,902.95.
SOLUTION We need to find i' such that $\$100,000 = \$16,902.95\, a_{\overline{10}|i'}$, which results in $i' = 0.1089$. □

3.3.1 Sinking-Fund Method Schedule

The level annual deposit into the sinking fund is $L/s_{\overline{n}|j}$. The accumulated value in the sinking fund after t deposits is $\big(L/s_{\overline{n}|j}\big)\, s_{\overline{t}|j}$. The outstanding balance at time t is the **net amount of the debt**, $OB_t = L - \big(L/s_{\overline{n}|j}\big)\, s_{\overline{t}|j}$. The principal repaid in the tth period is

$$PR_t = OB_{t-1} - OB_t = L\Big[\frac{s_{\overline{t}|j} - s_{\overline{t-1}|j}}{s_{\overline{n}|j}}\Big] = \frac{L(1+j)^{t-1}}{s_{\overline{n}|j}}.$$

The interest paid every year to the lender is $I = L\,i$. The interest earned in the sinking-fund account at time t is $\big(L/s_{\overline{n}|j}\big)\, s_{\overline{t-1}|j}\, j$. The **net interest** is the difference

$$I_t = L\,i - \big(L/s_{\overline{n}|j}\big)\, s_{\overline{t-1}|j}\, j = L\Big[i - \frac{(1+j)^{t-1} - 1}{s_{\overline{n}|j}}\Big].$$

Note that $I_t + PR_t = L\Big[i + \frac{1}{s_{\overline{n}|j}}\Big]$ is the borrower's total periodic outlay.

EXAMPLE 3.6(continued...) $L = 100,000$, $i = 0.1$, $j = 0.08$, $n = 10$. The annual payment of interest to the lender is $\$100,000\,(0.1) = \$10,000$. The annual level deposit to the sinking-fund account is $\$100,000/s_{\overline{10}|0.08} =$

$\$6,902.95$. The payment schedule is as follows

Time (in years)	Net Interest Due	Principal Repaid	Outstanding Balance
0	–	–	$\$100,000$
1	$10,000	$6,902.95	$100,000-$6,902.96 =$93,097.05
2	$10,000-$6,902.95(0.08) =$9447.76	$6,902.95(1+0.08) =$7,455.19	$93,097.05-$7,455.19 =$85,641.86
3	$8851.35	$8051.60	$ 77590.26
4	$8207.22	$ 8695.73	$68894.53
5	$7511.56	$9391.39	$59503.15
6	$6760.25	$10142.70	$ 49360.45
7	$5948.84	$10954.11	$ 38406.33
8	$5072.51	$11830.44	$ 26575.89
9	$4126.07	$12776.88	$ 13799.01
10	$3103.92	$13799.03	$ -0.02

□

4.1 Determination of Bond Prices

Definition A **bond** is a formal contract to repay borrowed money with interest at fixed intervals. Government or a corporation may issue bonds to raise funds to cover planned expenditures.

Definition Bonds issued by corporations are usually backed by various corporate assets as collateral. A **junk bond** is a bond that has little or no collateral. For example, a junk bond may be issued to raise funds to finance the takeover of another company.

Definition A **coupon** is the interest rate that the issuer pays to the bond holders.

Definition The **maturity date** is the date on which the bond issuer has to repay the principal amount. As long as all payments have been made, the issuer has no more obligation to the bond holders after the maturity date.

Another type of debt security is a **stock**, but the major difference between a stock and a bond is that stockholders have an equity stake in the company (i.e., they are owners), whereas bondholders have a creditor stake in the company (i.e., they are lenders). Another difference is that bonds usually have a defined maturity date, whereas stocks may be outstanding indefinitely.

Definition The **straight-term bond** is a bond with regular payments of interest plus a single payment of principal at the end of the term. This type of bond is similar to a loan. A bond specifies a **face amount** (principal), a

coupon rate (interest rate), and a **maturity date** or **term to maturity** during which coupons are to be paid, and the **redemption amount** that is to be repaid on the maturity date. Typically the face amount and the redemption amount are the same.

Let F denote the **face value** of a bond (also called **par value**), and let r denote the coupon rate per coupon period (defined by the bond issuer). Typically coupons are paid semiannually, so the coupon period is 6 months (unless stated otherwise). Let C be the redemption amount on the bond. It is typically equal to F, unless stated otherwise). Denote by n the number of coupon periods until the term of maturity (or simply **term**) of the bond. We will use j to denote the six-month yield rate determined by the market forces (for example, an interest rate that a person buying bonds can get somewhere else).

Definition The **price** of a bond (or the **purchase price**) is the total amount that a bondholder will receive from the bond issuer.

The present value of the price P of the bond on the issue date is calculated as

$$P = C\,\frac{1}{(1+j)^n} + Fr\Big[\frac{1}{1+j} + \frac{1}{(1+j)^2} + \cdots + \frac{1}{(1+j)^n}\Big] = C\,v^n + Fra_{\overline{n}|j}$$

where $v = (1+j)^{-1}$ is the present value factor. It is usually the case that $C = F$, therefore, the bond price can be expressed as

$$P = Fv^n + Fra_{\overline{n}|j}\,.$$

Using the definition of the present value of an n-payment annuity-immediate $v^n = 1 - j\,a_{\overline{n}|j}$, we see that

$$P = F(1 - ja_{\overline{n}|j}) + Fra_{\overline{n}|j} = F\big[1 + (r-j)a_{\overline{n}|j}\big],$$

or, alternatively,

$$P = Fv^n + Fr\frac{1-v^n}{j} = K + \frac{r}{j}(F - K)$$

where $K = Cv^n$ is the present value of the redemption amount. This formula is known as **Makeham's Formula**.

Definition If $r > j$, then $P > F$ and the bond is said to be bought **at a premium**.
If $r = j$, then $P = F$ and the bond is said to be bought **at par**.
If $r < j$, then $P < F$ and the bond is said to be bought **at a discount**.

Example 4.1 A 10% bond with semiannual coupons has a face amount of \$1,000 and was issued on June 18, 1994. The first coupon was paid on December 18, 1994, and the bond has a maturity date on June 18, 2014.

(a) Find the price of the bond on its issue date using $i^{(2)} = 0.05$ or 5%.

Solution It is given that $F = C = \$1,000$, $r = 0.1/2 = 0.05$, $n = 40$ months, and $j = 0.05/2 = 0.025$. Thus,

$$P = Fv^n + Fra_{\overline{n}|j} = \$1,000\,(1+0.025)^{-40}$$

$$+(\$1,000)(0.05)\frac{1-(1+0.025)^{-40}}{0.025} = \$1,627.57.$$

Note that $r > j$, hence $P > F$. This means that the bondholder bought this bond at a premium.

(b) Find the price of the bond on its issue date using $i^{(2)} = 0.10$ or 10%.

Solution Now $j = 0.1/2 = 0.05 = r$, and thus,

$$P = \$1,000\,(1+0.05)^{-40} + (\$1,000)(0.05)\frac{1-(1+0.05)^{-40}}{0.05} = \$1,000.$$

That is, $P = F$ and the bond is bought at par.

(c) Find the price of the bond on its issue date using $i^{(2)} = 0.15$ or 15%.

Solution Now $j = 0.15/0.075 > 0.05 = r$.

$$P = \$1,000\,(1+0.075)^{-40} + (\$1,000)(0.05)\frac{1-(1+0.075)^{-40}}{0.075} = \$685.14.$$

That is, $P < F$ and the bond is bought at discount. □

4.1.3 Bond Prices Between Coupon Dates

Suppose we wish to find the purchase price P_t of a bond at time t, where $0 \le t \le 1$ coupon period. The price of the bond is found as the present value of all future payments (coupons and redemption). Define P_0 to be the price of the bond at the beginning of the considered coupon period. Then

$$P_t = P_0(1+j)^t\,.$$

The **purchase price** P_t is also called the **price-plus-accrued** of the bond.

Definition The **market price** of a bond is equal to the price-plus-accrued minus the fraction of the coupon accrued to time t,

$$Market\ Price_t = P_t - tFr = P_0(1+j)^t - tFr.$$

Note that the time t is expressed as fraction of a coupon period, that is,

$$t = \frac{\#\ of\ days\ since\ last\ coupon\ paid}{\#\ of\ days\ in\ the\ coupon\ period}\,.$$

EXAMPLE 4.2 (Refer to Example 4.1) A 10% bond with semiannual coupons has a face amount of $1,000 and was issued on June 18, 2004. The first coupon was paid on December 18, 2004. Find the purchase price and market price on August 1, 2014, if $i^{(2)} = 0.05$.

SOLUTION On the last coupon date, June 18, 2014, the purchase price of the bond was

$$P = P_0 = Fv^n + Fra_{\overline{n}|j} = \$1,000\,(1+0.025)^{-20}$$

$$+(\$1,000)(0.05)\frac{1-(1+0.025)^{-20}}{0.025} = \$1,389.73.$$

The number of days from June 18 to August 1 is 44, and the number of days in the coupon period from June 18 to December 18 is 183. Hence, $t = 44/183$. The purchase price of the coupon on August 1, 2014, is calculated as

$$P_t = P_0(1+j)^t = \$1,389.73(1+0.025)^{44/183} = \$1,398.00.$$

The market price is

$$Market\ Price_t = \$1,398.00 - (44/183)(\$1,000)(0.05) = \$1,385.98. \quad \square$$

4.2 AMORTIZATION OF A BOND

A bond can be viewed as a standard amortized loan. The bondholder can be thought of as a lender and the issuer of the bond as the borrower. The purchase price P is the loan amount, and the coupon and redemption payments are the loan payments made by the borrower. The amortization schedule for a bond is

Time	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	$P = F\big[1+(r-j)a_{\overline{n}|j}\big]$
1	Fr	Pj $= F\big[j+(r-j)(1-v^n)\big]$	$Fr - Pj$ $= F(r-j)v^n$	$P - F(r-j)v^n$ $= F + F(r-j)a_{\overline{n-1}|j}$
2	Fr	$F\big[j+(r-j)(1-v^{n-1})\big]$	$F(r-j)v^{n-1}$	$F + F(r-j)a_{\overline{n-2}|j}$
		$\cdots$	$\cdots$	
$n-1$	Fr	$F\big[j+(r-j)(1-v^2)\big]$	$F(r-j)v^2$	$F + F(r-j)a_{\overline{1}|j}$
n	$Fr+F$	$F\big[j+(r-j)(1-v)\big]$	$F\big[1+(r-j)v\big]$	0

EXAMPLE 4.4 A 10% bond with face amount $10,000 matures 4 years after issue. Construct the amortization schedule for the bond over its term for nominal annual yield rate of 8%.

Solution Given $r = 0.05, j = 0.04, F = 10,000$, and $n = 8$. The amortization schedule is

Time	Payment Amount	Interest Due	Principal Repaid	Outstanding Balance
0	–	–	–	\$10,673.27
1	\$500	\$426.93	\$73.07	\$10,600.21
2	\$500	\$424.01	\$75.99	\$10,524.21
3	\$500	\$420.97	\$79.03	\$10,445.18
4	\$500	\$417.81	\$82.19	\$10,362.99
5	\$500	\$414.52	\$85.48	\$10,277.51
6	\$500	\$411.10	\$88.90	\$10,188.61
7	\$500	\$407.54	\$92.46	\$10,096.15
8	\$10,500	\$403.85	\$10,096.15	\$0

□

5.1 Internal Rate of Return Defined

A general financial transaction involves a number of cash inflows and outflows at various points in time.

Definition The **internal rate or return** for a transaction is the interest rate at which the value of all cash inflows is equal to the value of cash outflows.

Suppose that a transaction consists of a single amount L invested at time 0, and several future payments $K_1, K_2, \ldots, K_n$ to be received at times $1, 2, \ldots, n$. Then the equation of value is

$$L = K_1 \frac{1}{1+i} + K_2 \frac{1}{(1+i)^2} + \cdots + K_n \frac{1}{(1+i)^n},$$

and there is a unique solution for i, provided that $L < \sum_{j=1}^{n} K_j$.

It is possible to extend the notion of internal rate of return to more complex transactions. Suppose that at times $0, 1, \ldots, n$, there are payments received of amounts $A_0, A_1, \ldots, A_n$ and payments disbursed of amounts $B_0, B_1, \ldots, B_n$. The net amount received at time k is $C_k = A_k - B_k$. It can be positive or negative. If there is no amount paid at time k, then we assume $A_k = 0$. If there is no disbursement at time k, then $B_k = 0$.

Definition Suppose that a transaction has net cashflows of amounts $C_0, C_1, \ldots, C_n$ at times $0, 1, \ldots, n$. The **internal rate of return** for this transaction is any rate of interest i satisfying the equation of value $\sum_{k=0}^{n} C_k \frac{1}{(1+i)^k} = 0$.

Example A person buys 1,000 shares of stock at \$5 per share and pays a commission of 2%. Six months later he receives a cash dividend (a share of profits received by a stockholder) of \$0.20 per share, which he immediately reinvests commission-free in shares at a price of \$4 per share. Six months after that he buys another 500 shares at a \$4.50 per share, and pays a commission of 2%. Six months later he receives another cash dividend of \$0.25 per share and sells his existing shares at \$5 per share, again paying 2% commission. Find the internal rate of return for the entire transaction in the form $i^{(2)}$.

SOLUTION At time $t_0 = 0$, $A_0 = 0$ and $B_0 = \$5,100$. Six months later, at time $t_1 = 1$, $A_1 = \$200$ and $B_1 = \$200$, since he receives and immediately reinvests the dividend of \$200 buying additional 50 shares. At time $t_2 = 2$, $A_2 = 0$ and $B_2 = \$2,295$ (buying additional 500 shares and now owning a total of 1,550 shares). At $t_3 = 3$, $A_3 = \$387.50 + \$7,595 = \$7,982.50$ (the dividend on 1,550 shares plus the proceeds from the sale of the shares after the commission is paid), and $B_3 = 0$. The net amount received are $C_0 = -\$5,100$, $C_1 = 0$, $C_2 = -\$2,295$, and $C_3 = \$7,982.50$. The rate of return $i^{(2)}$ solves the equation

$$-5,100 - 2,295\frac{1}{(1+i^{(2)}/2)^2} + 7,982.5\frac{1}{(1+i^{(2)}/2)^3} = 0\,.$$

It can be found numerically that $i^{(2)} = 0.0649$ or 6.49%. □

5.1.4.1 PROFITABILITY INDEX

DEFINITION The **profitability index** is a ratio measuring the return per dollar of investment. It is

$$I = \frac{present\ value\ of\ cash\ inflows}{present\ value\ of\ cash\ outflow}\,.$$

EXAMPLE An investment of \$1,000 can be made into one of two projects. The first project generates income of \$250 per year for 5 years starting in one year, and the second project generates income of \$140 per year for 10 years. The shareholder's required return rate (called the **cost of capital**) is 5% per year for both projects. Compare the profitability indices for the two projects.

SOLUTION For Project 1, $I = \$250\,a_{\overline{5}|0.05}/\$1,000 = 1.0824$. For Project 2, $I = \$140\,a_{\overline{10}|0.05}/\$1,000 = 1.0810$. Project 1 has a higher profitability index and thus should be preferable to Project 2. □

5.1.4.2 PAYBACK PERIOD

DEFINITION Suppose an investment consists of a series of cash outflows $C_0, C_1, \ldots, C_t < 0$ followed by a series of cash inflows $C_{t+1}, C_{t+2}, \ldots, C_n > 0$. The **payback period** is the number of years required to recover the original amount invested, that is, it is the smallest k such that

$$-\sum_{s=0}^{t} C_s \leq \sum_{r=t+1}^{k} C_r\,.$$

Example In our previous example, for Project 1, $C_0 = \$1,000$, and $C_1 = C_2 = C_3 = C_4 = \$250$, thus the payback period is 4 years. For Project 2, the payback period is just over 7 years, (\$140)(7)=\$980. □

5.2.1 Dollar-Weighted Rate of Return

As a rule, the yield rate (or the return rate) of an investment is reported on an annual basis. There are two standard methods for measuring the yield rate: dollar-weighted and time-weighted rate of return.

Definition Suppose the following information is known:
(i) the balance in a fund at the start of a year is A,
(ii) for $0 < t_1 < t_2 < \cdots < t_n < 1$, the net deposit at time t_k is amount C_k, and
(iii) the balance in the fund at the end of the year is B.

Then the net amount of interest earned by the fund during the year is

$$I = B - \Big[A + \sum_{k=1}^{n} C_k\Big],$$

and the dollar-weighed rate of return earned by the fund for the year is the ratio

$$i = I/\Big[A + \sum_{k=1}^{n} C_k(1-t_k)\Big]\,.$$

The dollar-weighted rate of return solves the equation of value that equates the fund balance at the end of a year with all deposits minus all withdrawals, both accumulated to the end of the year with <u>simple</u> interest:

$$B = A(1+i) + C_1(1+i(1-t_1)) + C2(1+i(1-t_2)) + ... + C_n(1+i(1-t_n))\,.$$

Example 5.3 A pension fund began a year with a balance of \$1,000,000. There were contributions to the fund of \$200,000 at the end of February and again at the end of August. There was a benefit of \$500,000 paid out of the fund at the end of October. The balance remaining in the fund at the start of the next year was \$1,100,000. Find the dollar-weighted annual rate of return earned by the fund.

Solution Let i be the unknown rate of return. The equation of value for the dollar-weighted return is

$$\$1,000,000(1+i) + \$200,000(1+10/12\,i) + \$200,000(1+4/12\,i)$$

$$-\$500,000(1 + 2/12\,i) = \$1,100,000\,.$$

Solving for i, we get

$$i = \frac{\$1,100,000 + \$500,000 - \$1,000,000 - \$200,000 - \$200,000}{\$1,000,000 + \$200,000(10/12) + \$200,000(4/12) - \$500,000(2/12)} = 0.1739.$$

Note that if compound interest were used, then the equation of value would be

$$\$1,000,000(1+i) + \$200,000(1+i)^{10/12} + \$200,000(1+i)^{4/12}$$

$$-\$500,000(1+i)^{2/12} = \$1,100,000\,.$$

The numeric solution is $i = 0.1740$ which is very close to 0.1739. Note that since Taylor's expansion of $(1+i)^x$ is $1+xi+o(x)$, the formula for simple interest approximates that for compound one and gives a good approximation. □

5.2.2 Time-weighted Rate of Return

Definition Suppose the following information is known:
(i) the balance in a fund at the start of a year is A,
(ii) for $0 < t_1 < t_2 < \cdots < t_n < 1$, the net deposit at time t_k is amount C_k,
(iii) the value of the fund just before the net deposit at time t_k is F_k, and
(iv) the balance in the fund at the end of the year is B.

Then the time-weighed rate of return earned by the fund for the year is

$$\left[\frac{F_1}{A} \times \frac{F_2}{F_1 + C_1} \times \frac{F_3}{F_2 + C_2} \times \cdots \times \frac{F_k}{F_{k-1} + C_{k-1}} \times \frac{B}{F_k + C_k}\right] - 1\,.$$

Note that in this definition, the time length $t_k - t_{k-1}$ of each piece of the year is irrelevant. The ratio $F_j/(F_{j-1} + C_{j-1})$ is the **growth factor** for the period from t_{j-1} to t_j. The time-weighted return is found by compounding the successive growth factors over the course of the year.

Example 5.4 The following are the pension fund values after every transaction and at the end of the year. Find the time-weighted rate of return.

Fund Values	
Date	Amount
1/1/09	$1,000,000
3/1/09	$1,240,000
9/1/09	$1,600,000
11/1/09	$1,080,000
1/1/10	$900,000

Deposits	
2/28/09	$200,000
8/31/09	$200,000

Withdrawals	
10/31/09	$500,000
12/31/09	$200,000

SOLUTION The fund's earned rates for various time interval are:

1/1/09 to 2/28/09: $(\$1,240,000 - \$200,000 - \$1,000,000)/\$1,000,000 =$ net interest/starting amount $= 0.04$,
3/1/09 to 8/31/09: $(\$1,600,000 - \$200,000 - \$1,240,000)/\$1,240,000 = 0.129$,
9/1/09 to 10/31/09: $(\$1,080,000 + \$500,000 - \$1,600,000)/\$1,600,000 = -0.0125$,
11/1/09 to 12/31/09: $(\$900,000 + \$200,000 - \$1,080,000)/\$1,080,000 = 0.0185$.

The time-weighted rate of return for 2009 is $(1+0.04)(1+0.129)(1-0.0125)(1+0.0185) - 1 = 0.1809$. It can also be formulated as (see the definition)

$$\left(\frac{\$1,040,000}{\$1,000,000}\right)\left(\frac{\$1,400,000}{\$1,240,000}\right)\left(\frac{\$1,580,000}{\$1,600,000}\right)\left(\frac{\$1,100,000}{\$1,080,000}\right) - 1 = 0.1809. \quad \square$$

2.4.1 Yield Rates

DEFINITION Suppose an amount L is invested for an n year period and the value of the investment at the end of n years is M. Then the **annual yield rate** earned by this investment is the rate i that satisfies the equation $L(1+i)^n = M$.

EXAMPLE Consider a ten-year loan of $10,000 at the annual interest rate $i = 0.05$. We look at three ways at which the loan can be repaid:

(a) Ten annual level payments of $\$10,000/a_{\overline{10}|0.05} = \$1,295.05$. If these payments are immediately reinvested at the 5% interest rate, the accumulated value at the time of the last payment is $\$1,295.05\, s_{\overline{10}|0.05} = \$16,288.95$. Note that $\$10,000(1+0.05)^{10} = \$16,288.95$, so the lender realizes an annual yield rate of 5%. Algebraically, the amount $L/a_{\overline{n}|i}$ is reinvested at $i\%$ and the accumulated value is

$$\frac{L}{a_{\overline{n}|i}}\, s_{\overline{n}|i} = L\, \frac{(1+i)^n - 1}{i\big(1-(1+i)^{-n}\big)/i} = L\,(1+i)^n.$$

(b) Ten level annual interest payments of \$500 (5% interest), plus the payment of the \$10,000 principal at the end of ten years. If the payments of \$500 are immediately reinvested at 5%, the accumulated value after 10 years is $\$500\, s_{\overline{10}|0.05} = \$6,288.95$. Thus, along with the payment of \$10,000 principal at time 10, the total accumulated value is again \$16,288.95, which indicates the annual yield rate of 5%. Algebraically, the amount Li is reinvested at $i\%$ and the accumulated value is

$$Lis_{\overline{n}|i} + L = L\Big(i\frac{(1+i)^n - 1}{i} + 1\Big) = L(1+i)^n.$$

(c) A single payment of $\$10,000(1+0.05)^{10} = \$16,288.95$ at the end of 10 years. The lender receives an annual yield rate of 5% (by definition).

Note that for the lender to realize an annual yield rate of i (the original loan interest rate) over the n-year period, the lender should reinvest also at $i\%$. Otherwise, the yield rate will be different from i. □

Remark The reinvestment interest rate is used in determination of the yield rate (as in examples above), but is irrelevant in determination of the internal rate of return, since the internal rate of return of a loan is the rate of interest of the loan satisfying the equation of value (the loan amount is equal to the present value of the loan payments).

6.1 Spot Rates of Return

Definition A **zero coupon bond** is a bond that has no coupons and has a single payment made on maturity date. It is also called a **pure discount bond**.

Example Find the purchase price of a 2-year zero coupon bond with the face value \$1,000 assuming the current annual interest rate of 4.5%.

Solution The semiannual effective interest rate is $j = 0.045/2 = 0.0225$. The purchase price of the bond is

$$P = F\left(\frac{1}{1+j}\right)^4 = \frac{\$1,000}{1.0225^4} = \$914.84.$$

Note that P is always less than F. That is why it is a pure discount bond. □

Definition The **term structure of interest rates** at the current point in time is the set of yield rates on zero coupon bonds of all maturities. That is, it is the set $\{s_0(t)\}_{t>0}$ where $s_0(t)$ is the annual effective yield rate as of time 0 for a zero coupon bond maturing at time t. The quantity $s_0(t)$ is called the **spot rate of interest** for a t-year maturity zero coupon bond. The purchase price at time 0 of a t-year zero coupon bond is $F(1 + s_0(t))^{-t}$.

Any set of future cashflows can be valued now using the term structure. Suppose that payments of amounts $C_1, C_2, \ldots, C_n$ are due in $t_1, t_2, \ldots, t_n$ time periods from now. The total present value of the series of cashflows (the purchase price) is

$$P = C_1(1 + s_0(t_1))^{-t_1} + \cdots + C_n(1 + s_0(t_n))^{-t_n}. \qquad (*)$$

Definition The **yield to maturity** of a bond with n payment periods is the rate j that satisfies the equation:

$$P = \left[Fv_j^n + Fra_{\overline{n}|j}\right]$$

where P is defined by (*).

Example 6.1 Suppose that the current term structure has the following yields on zero coupon bonds:

Term	1/2 Year	1 Year	1 1/2 Year	2-Year
Zero Coupon Bond Rate	8%	9%	10%	11%

Find the price per \$100 face amount and yield to maturity of each of the following 2-year bonds with semiannual coupons:
(i) zero coupon bond, and (ii) 5% annual coupon bond.

SOLUTION (a) The price is $\$100(1+0.11/2)^{-4} = \$80.72$ and the annual yield rate is 11%, because $r=0$ and the annual yield rate j satisfies (*)

$$80.72 = 100\big(1+j/2\big)^{-4}.$$

(b) The price is

$$\$100(0.05/2)\Big[\frac{1}{1+0.08/2}+\frac{1}{(1+0.09/2)^2}+\frac{1}{(1+0.10/2)^3}\Big]$$

$$+\$100(1+0.05/2)\frac{1}{(1+0.11/2)^4} = \$89.59.$$

The annual yield rate j satisfies (*) ($r = 0.05/2 = 0.025$)

$$\$89.59 = \$100\frac{1}{(1+j/2)^4} + (\$100)(0.025)\frac{1-(1+j/2)^{-4}}{(j/2)},$$

or $j = 0.109354$. □

6.1 Spot Rates of Return

DEFINITION A **zero coupon bond** is a bond that has no coupons and has a single payment made on maturity date. It is also called a **pure discount bond**.

EXAMPLE Find the purchase price of a 2-year zero coupon bond with the face value \$1,000 assuming the current annual interest rate of 4.5%.

SOLUTION The semiannual effective interest rate is $j = 0.045/2 = 0.0225$. The purchase price of the bond is

$$P = F\left(\frac{1}{1+j}\right)^4 = \frac{\$1,000}{1.0225^4} = \$914.84.$$

Note that P is always less than F. That is why it is a pure discount bond. □

DEFINITION The **term structure of interest rates** at the current point in time is the set of yield rates on zero coupon bonds of all maturities. That is, it is the set $\{s_0(t)\}_{t>0}$ where $s_0(t)$ is the annual effective yield rate as of time 0 for a zero coupon bond maturing at time t. The quantity $s_0(t)$ is called the **spot rate of interest** for a t-year maturity zero coupon bond. The purchase price at time 0 of a t-year zero coupon bond is $F(1+s_0(t))^{-t}$.

Any set of future cashflows can be valued now using the term structure. Suppose that payments of amounts $C_1, C_2, \ldots, C_n$ are due in $t_1, t_2, \ldots, t_n$ time periods from now. The total present value of the series of cashflows (the purchase price) is

$$P = C_1(1+s_0(t_1))^{-t_1} + \cdots + C_n(1+s_0(t_n))^{-t_n}. \qquad (*)$$

Definition The **yield to maturity** of a bond with n payment periods is the rate j that satisfies the equation:

$$P = \left[Fv_j^n + Fra_{\overline{n}|j}\right]$$

where P is defined by (*).

Example 6.1 Suppose that the current term structure has the following yields on zero coupon bonds:

Term	1/2 Year	1 Year	1 1/2 Year	2-Year
Zero Coupon Bond Rate	8%	9%	10%	11%

Find the price per \$100 face amount and yield to maturity of each of the following 2-year bonds with semiannual coupons:
(i) zero coupon bond, and (ii) 5% annual coupon bond.

Solution (a) The price is $\$100(1+0.11/2)^{-4} = \$80.72$ and the annual yield rate is 11%, because $r=0$ and the annual yield rate j satisfies (*)

$$80.72 = 100\big(1+j/2\big)^{-4}.$$

(b) The price is

$$\$100(0.05/2)\left[\frac{1}{1+0.08/2}+\frac{1}{(1+0.09/2)^2}+\frac{1}{(1+0.10/2)^3}\right]$$

$$+\$100(1+0.05/2)\frac{1}{(1+0.11/2)^4} = \$89.59.$$

The annual yield rate j satisfies (*) ($r = 0.05/2 = 0.025$)

$$\$89.59 = \$100\frac{1}{(1+j/2)^4} + (\$100)(0.025)\frac{1-(1+j/2)^{-4}}{(j/2)},$$

or $j = 0.109354$. □

6.3.1 Forward Rates of Interest

Example Suppose that the current term structure has spot rates of $s_0(1) = 0.08$ for a one-year maturity and $s_0(2) = 0.09$ for a two-year maturity. If a person invests today \$1,000 in a one-year zero coupon bond, this amount will grow to \$1,080 in one year. If \$1,000 is invested in a two-year zero coupon bond, at maturity date the amount is $\$1,000(1.09)^2 = \$1,188.10$.

Suppose a person is able to borrow money at 8% annual interest, then the following situation is possible: the person borrows \$1,000 today and is committed to return \$1,080 in one year. He then invests the borrowed \$1,000 in a two-year zero coupon bond which will pay \$1,188.10 two years from now. The person's net outlay today is \$0 (because he invests borrowed money), and in one year, the net outlay is \$1,080. At the end of two years, however, the person receives $1,188.10. Thus, the whole transaction is actually a deferral by one year of a one-year investment of $1,080 with the annual return rate i that satisfies $\$1,080(1+i) = \$1,188.10$, that is, $i = 1,188.10/1,080 - 1 = 0.100092593$ or 10.01%. This rate is called a **one-year forward rate of interest**.

Another transaction is possible: the person borrows $1,000 today at 9% annual effective interest rate, so he is committed to pay $1,188.10 at the end of two years. He then invests $1,000 in a one-year zero coupon bond and receives $1,080 in one year. This transaction is equivalent to a one-year loan that is deferred by one year and has an effective annual interest rate 10.01%. □

The general definition of a forward rate of interest is as follows.

Definition Let $\{s_0(t)\}_{t>0}$ denote the term structure of zero coupon bond yield rates. The **$n-1$-year forward, one year interest rate for the year from time $n-1$ to time n** is

$$\frac{(1+s_0(n))^n}{(1+s_0(n-1))^{n-1}} - 1\,.$$

Note. "Forward rate" refers to a transaction that starts in the future, not now.

6.3.2 Arbitrage with Forward Rates of Interest

Definition An **arbitrage** is a possibility of risk-free profit at zero cost.

Example Suppose in our example, a lender offers to lend the person money for one year at rate 9.5% starting one year from now. Then to gain a positive profit with investment of amount 0, the person should do the following. He borrows $1,000 at the annual effective interest rate 8%, and is committed to repay $1,080 in one year. He invests $1,000 in a two-year zero coupon bond and receives $1,188.10 in two years. At the end of one year, the person receives $1,080 from the lender and repays his loan. At the end of two years, his net profit is $1,188.10-($1,080)(1.095)=$1,188.10-$1,182.60=$5.50.

Another arbitrage opportunity arises when a borrower offers to pay the person an interest rate of 11% for a one year investment starting one year from now. Then the person borrows $1,000 today at 9% annual effective interest rate and is committed to repay $1,881.10 at the end of two years. He then invests $1,000 in a one-year zero coupon bond and receives $1,080 in one year. He lends that amount to the borrower at rate 11% for one year. The person has made a net investment of zero, and at the end of two years his profit is $1,080(1.11)-$1,188.10=$1,198.80-$1,188.10=$10.70. □

6.4.1 Sport Betting Arbitrage

Example A person gambles online for sport events. Suppose he bets $1,000 on an event where team A plays with team B. Suppose also that one site offers a $2.50 return on a bet of $1 if team A wins, while another site offers a $2.50 return on a bet of $1 if team B wins. If the person bets $500 on each site, his net profit is $500 * 2.5 - \$1,000 = \$1,250 - \$1,000 = \$250$.

More generally, if one site offers $m for a bet of $1 if team A wins, and another site offers $n for a bet of $1 if team B wins, then the person should bet $\$x$ on the first site and $\$y$ on the second site, where x and y satisfy

$$\begin{cases} x + y = 1,000 \\ xm = yn \end{cases} \leftrightarrow \begin{cases} x = \frac{1,000n}{m+n} \\ y = \frac{1,000m}{m+n} \end{cases}$$

The person's net profit is $xm-1,000. For example, if one site offers $2 for a $1 bet if team A wins, and another site offers $3 for a $1 bet if team B wins, then a gambler should bet $x=$1,000(3)/(2+3)=$600 for team A, and $400 for team B. Then the net profit is $1,200-$1,000=$200. □

The website for sport betting arbitrage opportunities is www.arbhunters.co.uk.

8.1 The Dividend Discount Model of Stock Valuation.

Definition A **stock** represents ownership in a corporation, entitling the stockholder to certain privileges including the voting rights. In addition, stockholders receive periodically **dividends** which are shares of profit earned by the corporation. Selling shares of stock is the way for the corporation to raise funds.

A person buying stock shares would be looking for a return on the investment in the form of future dividends and share price increases. For an investor with a long-term outlook, intending to hold the stock indefinitely, the price of the stock might be regarded as the present value of future dividends expected to be paid on the stock.

If d_t denotes the expected dividend payable at the end of the tth year from the stock purchase date, and let i denote the annual effective rate of return. Then the price of the stock can be computed according to the **dividend discount model for valuing stocks**,

$$P = \sum_{t=1}^{\infty} \frac{d_t}{(1+i)^t}.$$

Another formulation is in terms of the spot rates $s_0(t)$,

$$P = \sum_{t=1}^{\infty} \frac{d_t}{(1+s_0(t))^t}.$$

EXAMPLE 8.1 A stock is expected to pay dividends at the end of each year indefinitely. An investor wishes to receive an effective annual rate of return of 5%. Find the stock price if
(a) dividends are level at $2 per year.
SOLUTION The price is a present value of a perpetuity-immediate. Using the fact that

$$\sum_{t=1}^{\infty} a^t = a + a^2 + a^3 + ... = a(1 + a + a^2 + ...) = \frac{a}{1-a},$$

we see that the price is

$$P = \sum_{t=1}^{\infty} \frac{\$2}{(1+0.05)^t} = \frac{\$2(1.05)^{-1}}{1-(1.05)^{-1}} = \frac{\$2}{1.05-1} = \frac{\$2}{0.05} = \$40.$$

(b) The first dividend is $2, and subsequent dividends increase by 2% every year.
SOLUTION The price is

$$P = \sum_{t=1}^{\infty} \frac{\$2(1+0.02)^{t-1}}{(1+0.05)^t} = \frac{\$2}{1.02} \sum_{t=1}^{\infty} \left(\frac{1.02}{1.05}\right)^t$$

$$= \frac{\$2}{1.02} \frac{1.02/1.05}{1-1.02/1.05} = \frac{\$2}{1.05-1.02} = \frac{\$2}{0.03} = \$66.67.$$

8.2 SHORT SALE OF STOCK.

Typically an investor buys a stock and sells later at a higher price. An alternative way is to sell high a stock that an investor doesn't own (stock is borrowed from a lender) and then buy at a lower price later (and returns the stock plus dividends). This way is called a **short sale** of the stock.

Suppose the price of the stock is S_0 at the time the short sale is opened, and is S_1 at the closure time. If the stock paid a dividend of amount D, then the short seller must pay that amount to the original owner of the stock, and therefore, the net gain made by the short seller is $S_0 - S_1 - D$.

In practice, this transaction is arranged through an investment dealer. An initial amount called **margin** is required from the short seller at the time the short sale is made. The margin may be up to 50% of the stock price. The investor opens an account called the **margin account**. It is owned by the investor but administered by the dealer. If dividends are paid on the stock, they are deducted from the margin account. The money in the margin account is owned by the investor, and the dealer may pay interest on the account. A short sale can be summarized as follows: (i) stock price at time 0 is S_0, (ii) margin required at time 0 is M, (iii) margin account pays interest at rate i per period, (iv) stock pays dividend of amount D at time 1, and (v) stock price at time 1 is S_1.

The short seller opens the margin account at time 0 with a deposit of amount M. The short sale is initiated at time 0. At time 1, the amount M grows with interest to $M(1+i)$. Also at time 1, when the short sale is terminated, the investor must pay S_1 to buy the stock. The net gain on the short sale is $S_0 - S_1$. The net amount on the margin account at time 1, after the short sale is closed and the dividend is paid, is $M(1+i)+S_0-S_1-D$. The actual return rate j earned by the investor satisfies the equation

$$M(1+j) = M(1+i) + S_0 - S_1 - D.$$

Example 8.2 The margin requirement on a short sale of stock is 50% of the stock price, and the margin account pays 10% per year. The price of the stock is \$100 at the start of a year when the stock is sold short, and the short sale is closed at the end of the year and a dividend of \$5 is paid. Find the return rate earned by the short seller if the stock price at the end of the year is (a) \$90.
Solution Given $M = \$100(0.5) = \$50, i = 0.1, S_0 = \$100, S_1$ =90, D=\$5. The return rate satisfies

$$\$50(1+j) = \$50(1+0.1) + \$100 - \$90 - \$5,$$

thus, $j = 0.2$ or 20%.
(b) \$100.
Solution The return rate satisfies

$$\$50(1+j) = \$50(1+0.1) + \$100 - \$100 - \$5,$$

thus, $j = 0$ or 0%.
(c) \$110.
The return rate satisfies

$$\$50(1+j) = \$50(1+0.1) + \$100 - \$110 - \$5,$$

thus, $j = -0.2$ or -20%.

9.1 Forward Contract

Definition A **forward contract** is an arrangement to buy or sell a certain asset at a specific future time called the **delivery date**, for a specific price called the **delivery price**. No money changes hands at time 0, and the transaction takes place at the time T. The side that pays the delivery price at time T is said to **have a long position on the forward contract**, and the side that delivers the asset and is paid at time T is said to **have a short position on the forward contract**.

Denote by S_0 the value of the asset at time 0 (called the **spot price at time 0**), and let S_T be the value of the asset at time T (called **spot price at time** T). Let $F_{0,T}$ denote the delivery price. The **payoff on the long position** at time T is $S_T - F_{0,T}$. The **payoff on the short position** at time T is $F_{0,T} - S_T$.

Example 9.1 A corporation enters into a forward contract with a gold refiner to purchase 1,000 ounces of gold in one year (at time $T = 1$) at the delivery price of $950 per ounce. Suppose that the spot price at time $T = 1$ of gold is $900 per ounce. Find the payoffs on the long and short positions.

Solution The payoff on the long position held by the corporation is $S_1 - F_{0,1} = 1,000(\$900 - \$950) = -\$50,000$. The payoff on the short position held by the gold refiner is $50,000. Note that the corporation is obliged by the forward contract to pay $950,000 for the gold that is worth $900,000 in the current market. □

9.1.2 Prepaid Forward Price on an Asset Paying No Income

Definition A **prepaid forward contract** is an arrangement to buy or sell a certain asset, in which the side with the long position (called the **long side**) pays the price at time 0, and the side with the short position (called the **short side**) delivers the asset at time T. The amount paid by the long side at time 0 is called the **prepaid forward price**.

Denote by $F^P_{0,T}$ the prepaid forward price. For an asset that pays no dividends or coupons or any other income, the prepaid forward price is equal to the spot price at time 0, that is, $F^P_{0,T} = S_0$. If $F_{0,T}$ is not equal to S_0, then it is possible to create an arbitrage opportunity.

Example 9.2 A share of stock costs $100 at time 0. Suppose that a prepaid forward price is $105. How to make an arbitrage gain?

Solution Sell the forward contract at $105 with the delivery date in one year. Buy a share of stock for $100, and put the $5 into your pocket. □

10.4.1. Pricing Stock Options. An example.

Definition. A stock is a share in a company. The price of the stock changes randomly.
Definition. A bond is an interest-bearing certificate issued by the government or business, and redeemable on a specified date. The price of the bond is fixed.
Definition. An option is a right to purchase a stock at a future time at a fixed price. Note that the option is a right, but not an obligation.
Definition. An investor's portfolio is a collection of shares of a risky stock,

options to buy a stock, and secure bonds.

Example (Option Pricing). Suppose we have an option to purchase a stock at a future time at a fixed price. How much should we pay for this option now?
Suppose that the present value of a stock is \$100 per share. After one time period it will be worth either \$200 or \$50. We are given an option, at a cost of cy, to buy y shares of this stock in one time unit for \$150 per share. So, if we purchase the option, and the stock rises to \$200, we will exercise the option and get \$50 per share of a net profit. However, if the stock is worth \$50, we will not exercise the option.
Besides the option, we purchase for our portfolio x shares of the stock for \$100 each. We will assume that both x and y can be positive as well as negative (negative meaning we are selling the stock or option).
We are interested in determining c, the unit cost of the option. We will show that unless $c = 50/3$, there will be a portfolio that will result in a positive gain.

Our portfolio at time 1 will be worth \$200x+\$50y if the stock price is \$200, and \$50x if the stock price is \$50. Suppose we choose y so that $200x + 50y = 50x$, or $y = -3x$. Therefore, the cost of the original portfolio is $100x - 3xc$, and the gain is $50x - 100x + 3xc = (3c - 50)x$ which is zero if $c = 50/3$ and can be made positive otherwise. For example, if $c = 20$, then we buy one share of stock (x=1) and sell three shares of the option (y=-3) and make the profit of \$10. If $c = 15$, then we sell one share of stock (x=-1) and buy three shares of options (y=3), and attain the profit of \$5.
Definition. A sure win betting scheme is called an arbitrage.
In this example, there is no arbitrage only if $c = 50/3$.

10.4.2. The Arbitrage Theorem.

Consider an experiment with the sample space $S = \{1, 2, \ldots, m\}$. Suppose that n wagers are available. A betting scheme is a vector $\mathbf{x} = (x_1, \ldots, x_n)$ such that x_1 is bet on wager 1, X_2 is bet on wager 2,..., and x_n is bet on wager n. If the outcome of the experiment is j, then the return from the betting scheme is $\sum_{i=1}^{n} x_i\, r_i(j)$.
Theorem (The Arbitrage Theorem). If X is the outcome of the experiment, then there is a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ for X such that for all $i = 1, \ldots, n$, $\mathbb{E}[r_i(X)] = \sum_{j=1}^{m} p_j\, r_i(j) = 0$. Else, there is a betting scheme that leads to a sure win.
Example. In our example, there are two possible outcomes – the values of the stock at time 1 – \$200 and \$50. There are two wagers: to buy (sell) the stock, and to buy (sell) the option. By the arbitrage theorem, there is no sure win strategy if there exists $(p, 1-p)$ such that the expected return under both wagers is 0. The return for stock is $200 - 100 = 100$ if the outcome is 200, and $50 - 100 = -50$ if the outcome is 50. Thus, $\mathbb{E}[$stock return$] = 100\,p - 50(1-p) = 0$ iff $p = 1/3$. Then, the expected return for option is $\mathbb{E}[$ option return$] = (50-c)p - c(1-p) = 50p - c = 50/3 - c = 0$ iff $c = 50/3$.

10.4.3. The Black-Scholes Option Pricing Formula.

Definition. Suppose we will be given amount $\$v$ at time t. If we were given $\$v$ now, we could've loaned out the money with interest at a continuously compounded rate of $100\,\alpha\%$ per unit time, and get $\$v\, e^{\alpha t}$ at time t. Therefore, the present value of $\$v$ given to us at time t is $v\, e^{-\alpha t}$. The quantity α is called the discount factor. The function $e^{-\alpha t}$ is called the discount function.

Derivation of the Black-Scholes (Merton) Model (1973).

Let $X(t)$ be the price of a stock at time t. The present value of the stock is $X(0) = x_0$.
Suppose we have two wagers. One is to buy (or sell) the stock at the price $X(s)$ at time $s < t$, and then sell (or buy) this stock at time t for the price $X(t)$. The other wager is to buy (or sell) an option that gives us the right to buy stock at time t for a price K per share.
To deal with the first wager, compute the present values of the stock prices $X(s)$ and $X(t)$, so that they are on the same scale. The present value of the stock price at time s is $e^{-\alpha s}\, X(s)$, and the present value of the stock price at time t is $e^{-\alpha t}\, X(t)$.
By the arbitrage theorem, there is no sure win strategy if the expected return of this wager is zero, that is,

$$\mathbb{E}[e^{-\alpha t}\, X(t) \,|\, X(u),\, 0 \le u \le s] = e^{-\alpha s}\, X(s). \quad (*)$$

Now we go to the second wager. The option is worth $(X(t) - K)^+$ at time

t, hence, and the present value of the option is $e^{-\alpha t}(X(t)-K)^+$. If c is the cost of the option, then, by the arbitrage theorem,

$$\mathbb{E}[e^{-\alpha t}(X(t)-K)^+ - c] = 0. \quad (**)$$

To find c, we need to find a probability that satisfies (*). Then, c will satisfy (**) computed under the same probability.

The Black-Scholes model assumes that $X(t)$ is a geometric Brownian motion, that is, $X(t) = x_0\, e^{Y(t)}$ where $Y(t) \sim N(\mu\, t, \sigma^2\, t)$. Therefore,

$$\mathbb{E}\Big[X(t)\,|\,X(u),\,0 \le u \le s\Big] = \mathbb{E}\Big[x_0\, e^{Y(t)}\,|\,Y(u),\,0 \le u \le s\Big]$$
$$= x_0\,\mathbb{E}\Big[e^{Y(t)-Y(s)+Y(s)}\,|\,Y(u),\,0 \le u \le s\Big]$$
$$= x_0\, e^{Y(s)}\,\mathbb{E}\Big[e^{Y(t)-Y(s)}\,|\,Y(u),\,0 \le u \le s\Big]$$
$$= X(s)\mathbb{E}\Big[e^{Y(t)-Y(s)}\Big] = X(s)\, e^{\mu(t-s)+\sigma^2\,(t-s)/2}.$$

Let $\mu + \sigma^2/2 = \alpha$. Then,

$$\mathbb{E}[e^{-\alpha\, t}\, X(t)\,|\,X(u),\,0 \le u \le s] = e^{-\alpha\, t}\, X(s)\, e^{\alpha\,(t-s)} = e^{-\alpha\, s}\, X(s).$$

Thus, the equation (*) is satisfied if we choose the probability corresponding to $X(t) = x_0\, e^{Y(t)}$ where $Y(t) \sim N(\mu\, t, \sigma^2\, t)$, and where $\mu + \sigma^2/2 = \alpha$.
Now we can compute c from equation (**). We have

$$c = \mathbb{E}\Big[e^{-\alpha\, t}(X(t)-K)^+\Big] = e^{-\alpha\, t}\int_{-\infty}^{\infty}(x_0\, e^y - K)^+\,\frac{1}{\sqrt{2\pi\,\sigma^2\, t}}\, e^{-(y-\mu\, t)^2/(2\sigma^2\, t)}\, dy$$
$$= \int_{\ln(K/x_0)}^{\infty}(x_0\, e^y - K)\,\frac{1}{\sqrt{2\pi\,\sigma^2\, t}}\, e^{-(y-\mu\, t)^2/(2\sigma^2\, t)}\, dy$$
$$\overset{\text{pg616}}{=\!=\!=} x_0\,\Phi(\sigma\,\sqrt{t}+b) - Ke^{-\alpha\, t}\Phi(b)$$

where

$$b = \frac{\mu\, t - \ln(K/x_0)}{\sigma\sqrt{t}}.$$

Example. The current price of a stock is \$100. Suppose the stock price can be modelled by the Black-Scholes model with drift coefficient $\mu = -0.45$ and volatility $\sigma = 1$. Compute the cost of the option to buy the stock at time $t = 3$ for the cost of $K = \$100$.
Solution: $\alpha = \mu + \sigma^2/2 = 0.05$, $b = \frac{(-0.45)(3)-\ln(100/100)}{(1)\sqrt{3}} \approx -0.78$, $c = (100)\Phi(0.95)-(100)e^{-(0.05)(3)}\Phi(-0.78) \approx (100)(0.8289)-(100)(0.8607)(0.2177) \approx$ \$64.17.

7.2 Asset-Liability Matching and Immunization.

In the course of conducing business, a company will make commitments involving future income and outgo of capital. To maintain viable (or even profitable) position, the company will make investments so that funds are available for outgoing payments whenever needed. Denote by L_t the company's liability due (or outgoing payments) at time t, and let A_t be the company's asset income (revenue, inflow of cash, proceeds) at time t. If the company can arrange its investments so that the asset income exactly covers the liability due at each point in time, that is, $L_t = A_t$ for all t, then the projected asset income and liabilities due are said to be exactly matched.

Example 7.5. A small company is terminating its operations and has decided to provide its three employers with severance package that pays \$10,000 per year (at the end of each year) up to and including age 65, and a lump sum of \$100,000 at age 65. The employees are now exact ages 50, 53, and 55. The company buys three bonds with a face amount of \$100,000 each, an annual coupon rate 10% and maturities of 10, 12, and 15 years. Determine the cost to the company to fund the severance packages if the bonds have effective annual yield rates of 10% for the 10-year bond, 11% for the 12-year bond, and 12% for the 15-year bond.

Solution: $P = Fv^t + Fra_{\overline{n}|j}$. For the 10-year package, $P = 100,000/(1.1)^{10} + 100,000(0.1)(1-(1.1)^{-10})/0.1 = 100,000$; for the 12-year package, $P = 100,000/(1.11)^{12} + 100,000(0.1)(1-(1.11)^{-12})/0.11 = 93,507.64$; and for the 15-year package, $P = 100,000/(1.12)^{15} + 100,000(0.1)(1-(1.12)^{-15})/0.12 = 86,378.27$. The total cost to the company is $279,885.91$. With purchase of these bonds, the company's liability to the three employees are exactly matched.

7.2.1. Redington Immunization.

In asset-liability matching we assumed that the term structure doesn't change over time, and that the yield rate i_0 is constant for all times to maturity. That is,

$$PV_A(i_0) = \sum_{t=0}^{n} A_t\, v_{i_0}^t = \sum_{t=0}^{n} L_t\, v_{i_0}^t = PV_L(i_0). \quad (*)$$

In reality, it is impossible to know the future behavior of interest rates i_0. Without exact matching ($A_t = L_t$ for all t), there is a risk that there won't be sufficient asset income to balance the liabilities due, that is, $PV_A(i) < PV_L(i)$.

In 1952, an actuary F.M. Redington introduced a theory of immunization for an asset/liability flow. According to this theory, with a careful structuring of A_t in relation to L_t, small variations in the interest rate will result in surplus, that is, $PV_A(i) > PV_L(i)$ for both $i > i_0$ and $i < i_0$. Suppose the allocation of asset income satisfies the following conditions:

$$PV_A(i_0) = PV_l(i_0), \quad \frac{d}{di}PV_A(i)\Big|_{i_0} = \frac{d}{di}PV_L(i)\Big|_{i_0}, \quad (**)$$

and

$$\frac{d^2}{di^2}PV_A(i)\Big|_{i_0} > \frac{d^2}{di^2}PV_L(i)\Big|_{i_0}. \quad (***)$$

Define the function $h(i) = PV_A(i) - PV_L(i)$. We have $h(i_0) = h'(i_0) = 0$ and $h''(i_0) > 0$. Hence, $h(i)$ has a local minimum at i_0, and thus, there exist an interval (i_l, i_u) such that for all i in this interval, $h(i) > h(i_0)$ or, equivalently, $PV_A(i) > PV_L(i)$. In this case the asset/liability flow is immunized against small changes in interest rate i. The immunization of the portfolio is called Redington immunization.

Example 7.7 To immunize liabilities due in the severance packages in our previous example, the company purchases an investment portfolio consisting of two zero coupon bonds, due at times t_1 and t_2. Suppose that the term structure is constant at an effective annual rate of 10%. For each of the following pairs t_1 and t_2, determine how many zero coupon bonds must be purchased and whether the portfolio is in an immunized position.

(a) $t_1 = 6$, $t_2 = 12$.
(b) $t_1 = 2$, $t_2 = 14$.

Solution: Let X and Y be the amounts of purchased zero coupon bonds with maturity at times t_1 and t_2, respectively. We need to find X and Y. Under (*),$X\,v^{t_1} + Y\,v^{t_2} = \sum_{t=1}^{15} L_t\,v^t = 300,000$, the present value of liabilities. Note that $v = 1/(1+i)$, therefore,

$$\frac{d}{d\,i}v^t = \frac{d}{d\,i}\frac{1}{(1+i)^t} = -t\frac{(1+i)^{t+1}}{=}(-v)\,(t\,v^t).$$

Using that and (**), we have $t_1\,X\,v^{t_1} + t_2\,Y\,v^{t_2} = \sum_{t=1}^{15} t\,L_t\,v^t = 30,000v + 2(30,000)v^2 + 3(30,000)v^3 + ... + 9(30,000)v^9 + 10(130,000)v^{10} + 11(20,000)v^{11} + 12(120,000)v^{12} + 13(10,000)v^{13} + 14(10,000)v^{14} + 15(110,000)v^{15} = 2,262,077.228$.

Solving these two equations for X and Y, we get

(a) $X = \$395,035.30$ and $Y = \$241,699.38$.
(b) $X = \$195,407.21$ and $Y = \$525,977.96$.

To see whether portfolio is immunized, check whether (***) holds. We have to check whether $t_1^2\,X\,v^{t_1} + t_2\,Y\,v^{t_2} > \sum_{t=1}^{15} t^2\,L_t\,v^t = 22,709,878.$.

(a) $t_1^2\,X\,v^{t_1} + t_2\,Y\,v^{t_2} = 19,117,390$, so not immunized.

(b) $t_1^2\, X\, v^{t_1} + t_2\, Y\, v^{t_2} = 27,793,236$, so immunized.

Hedging

Definition. Hedging is a strategy intended to protect an investment against loss. It involves buying securities that move in the opposite direction than the asset being protected.

Example. An investor believes that the stock price of company A will rise over the next month and wants to buy their stock. But there is a risk that stock prices might fall across the whole industry that company A is involved into. Since the investor is interested in the specific company, rather than the entire industry, he wants to hedge out the industry-related risk by short selling (selling stock that is not owned) an equal value of the shares of company A's direct competitor, company B. The first day the investor's portfolio is:

- Buys 1,000 shares of company A's stock at $1 each
- Has a short sale arranged for 500 shares of Company B at $2 each

On the second day, a favorable news story about the industry is published and the value of all stock goes up. Because company A is stronger, its stock price increases by 10%, while for company B, it increases by just 5%. The portfolio now is:

- Owns 1,000 shares of company A's stock at $1.10 each: $100 gain
- Short sale of 500 shares of Company B at $2.10 each: $50 loss

On the third day, an unfavorable news story is published about the health effects of the industry, and all stocks crash 50%.

The value of the portfolio over these three days is:

- Company A's stock Day 1: $1,000, Day 2: $1,100, Day 3: $550; net loss of $450.
- Company B's stock Day 1 -$1,000, Day 2: -$1,050, Day 3: -$525, net profit of $475.

Without the hedge, the investor would have lost $450 on stock of company A (or $900 if invested the $1,000 not in company B's stock but in additional shares of company A's stock). Here the hedge (the short sale of company B's stock) gives a profit of $475, for a net profit of $25 during a market collapse.