STAT 484 Actuarial Science: Models

INTRODUCTION

Who is an Actuary?

An actuary is a person who analyzes financial risks for different types of insurance and pension programs. The word "actuary" comes from the Latin for account keeper, deriving from *actus* "public business."

If you are seriously considering becoming an actuary, then visit

www.beanactuary.org

In the United States, actuaries have two professional organizations: the Society of Actuaries (SOA) and the Casualty Actuarial Society (CAS).

What is the Society of Actuaries (SOA)?

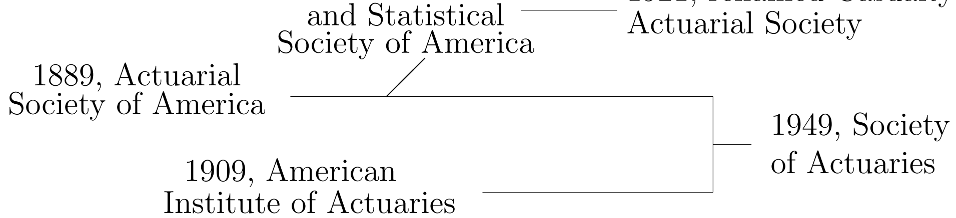
SOA members work in life and health insurances, and retirement programs. Check their website at www.soa.org

What is the Casualty Actuarial Society (CAS)?

CAS members work in property (automobile, homeowner) and casualty insurance (workers' compensation). Check their website at www.casact.org

Historical Background

The first actuary to practice in North America was Jacob Shoemaker of Philadelphia, a key organizer in 1809 of the Pennsylvania Company for Insurances on Lives and Granting Annuities. The Actuarial Society of America came into being in New York in 1889. In 1909, actuaries of life companies in the midwestern and southern United States organized the American Institute of Actuaries with headquarters in Chicago. In 1914 the actuaries of property and liability companies formed the Casualty Actuarial and Statistical Society of America which in 1921 was renamed the Casualty Actuarial Society. In 1949, the Society of Actuaries was created as the result of the merger between the Actuarial Society of America and the American Institute of Actuaries.



The information below is taken from The Society of Actuaries *Basic Education Catalog*, Spring 2006.

Anyone pursuing actuarial career may apply to the SOA. He would be a member of this organization as long as he pays the dues. Anyone who has passed certain number of actuarial exams and met some additional requirements, becomes an Associate of the Society of Actuaries (ASA), and then, after passing more exams and meeting some more additional requirements one becomes a Fellow of the Society of Actuaries (FSA).

What are Actuarial Exams?

Historical Note: In 1896, after some hesitation, an examination system was adopted. The first Fellow by examination qualified in 1900.

SOA currently offers nine actuarial exams.

Exam 1	Probability (same as SOA Exam P)
Exam 2	Financial Mathematics (same as SOA Exam FM)
Exam 3	Actuarial Models: Segment 3F, Financial Economics (same as SOA Exam MFE) and Segment 3L, Life Contingencies and Statistics
Exam 4	Construction and Evaluation of Actuarial Models (same as SOA Exam C)
Exam 5	Introduction to Property and Casualty Insurance and Ratemaking
Exam 6	Reserving, Insurance Accounting Principles, Reinsurance, and Enterprise Risk Management
Exam 7	Law, Regulation, Government and Industry Insurance Programs, and Financial Reporting and Taxation
Exam 8	Investments and Financial Analysis
Exam 9	Advanced Ratemaking, Rate of Return, and Individual Risk Rating Plans

In this course we review the material for Exam 1/P and learn roughly 25% of what is needed to pass Exam 3 Segment L. A disclaimer is in order here. Taking this course doesn't guarantee that you pass these exams. The course prepares you to start preparing for the exams.

Exam 1/P Probability

The examination for this material consists of three hours of multiple-choice questions offered through computer-based testing.
The purpose of this exam is to check the knowledge of the fundamental probability tools for quantitatively assessed risk. The application of these tools to problems encountered in actuarial science is emphasized. A thorough command of probability topics and the supporting calculus is assumed. Additionally, a very basic knowledge of insurance and risk management is assumed. A table of values for the normal distribution is included with the examination.

The exam covers the following probability topics in a risk management context:

1. General Probability
   - Set functions including set notation and basic elements of probability
   - Mutually exclusive events
   - Additive and multiplicative laws
   - Independence of events
   - Conditional probability
   - The Bayes Theorem

2. Univariate probability distributions (including binomial, negative binomial, geometric, hypergeometric, Poisson, uniform, exponential, chi-square, beta, Pareto, lognormal, gamma, Weibull, and normal)
   - Probability functions and probability density functions
   - Cumulative distribution functions
   - Mean, median, mode, percentiles, and moments
   - Variance and measures of dispersion
   - Moment generating functions
   - Transformations
   - Independence of random variables

3. Multivariate probability distributions (including the bivariate normal)
   - Joint probability functions and joint probability density functions
   - Joint cumulative distribution functions
   - Central Limit Theorem
   - Conditional and marginal probability distributions
   - Moments for joint, conditional, and marginal probability distributions
   - Joint moment generating functions
   - Covariance and correlation coefficient
   - Transformations and order statistics
   - Probabilities and moments for linear combinations of independent random variables

Risk and Insurance

Reference: *Risk and Insurance*, Study Notes, SOA web site, code P-21-05.

Definitions. People need economic security (food, clothing, shelter, medical care, etc.). The possibility to lose the economic security is called the economic risk or simply risk. This risk causes many people to buy insurance. Insurance is an agreement where, for a stipulated payment called the premium, one party called the insurer agrees to pay the other called the policyholder (or insured) or his designated beneficiary a defined amount called claim payment or benefit upon the occurrence of a specific loss. This defined claim payment can be a fixed amount or can reimburse all or a part of the loss that occurred. The insurance contract is called the policy. Only small percentage of policyholders suffer loss. Their losses are paid out of the premiums collected from the pool of policyholders. Thus, the entire pool compensates the unfortunate few. Each policyholder exchanges an unknown loss for the payment of a know premium.

The insurer considers the losses expected for the insurance pool and the potential of variation in order to charge premiums that, in total, will be sufficient to cover all of the projected claim payments for the insurance pool. The insurer may restrict the particular kinds of losses covered. A peril is a potential cause of a loss. Perils may include fires, hurricanes, theft, or heart attacks. The insurance policy may define specific perils that are covered, or it may cover all perils with certain exclusions such as, for example, property loss as a result of a war or loss of life due to suicide.

Losses depend on two random variables. The first is the number of losses that will occur in a specified period. This random variable is called frequency of loss and its probability distribution is called frequency distribution. The second random variable is the amount of the loss, given that a loss has occurred. This random variable is called the severity and its distribution is called the severity distribution. By combining the frequency and the severity distributions, one can determine the overall loss distribution.

Example. Suppose a car owner will have no accidents in a year with probability 0.8 and will have one accident with probability 0.2. This is the frequency distribution. Suppose also that with probability 0.5 the car will need repairs costing \$500, with probability 0.4 the repairs will cost \$5,000, and with probability 0.1 the car will need to be replaced at the cost \$25,000. This is the severity distribution. Combining the two distributions, we have that the distribution of X, the total loss due to an accident is

$$f(x) = \begin{cases} 0.8, & \text{if } x = 0 \\ (0.2)(0.5) = 0.1, & \text{if } x = 500 \\ (0.2)(0.4) = 0.08, & \text{if } x = 5,000 \\ (0.2)(0.1) = 0.02, & \text{if } x = 25,000. \end{cases}$$

Definitions (continued...) The expected amount of claim payments is called the net premium or benefit premium. The gross premium is the total of the net premium and the amount to cover the insurer's expenses for selling and servicing the policy, including some profit. Policyholders are willing to pay a gross premium for an insurance policy, which exceeds the expected amount of their losses, to substitute the fixed premium payment for a potentially enormous payment if they are not insured.

What kind of risk is insurable?

An insurable risk must satisfy the following criteria:
1. The potential loss must be significant so that substituting the premium payment for an unknown economic outcome (given no insurance) is desirable.
2. The loss and its economic value must be well-defined and out of the policyholder's control. For example, the policyholder should not be allowed to cause a loss or to lie about its value.
3. Covered losses should be reasonably independent. For example, an insurer should not insure all the stores in one area against fire.

Examples of Insurance.

1. the auto liability insurance – will pay benefits to the other party if a policyholder causes a car accident.
2. the auto collision insurance – will pay benefits to a policyholder in case of a car accident.
3. the auto insurance against damages other than accident – will pay benefits to a policyholder in case the car is damaged from hailstones, tornado, vandalism, flood, earthquake, etc.
4. the homeowners insurance – will pay benefits to a policyholder towards repairing or replacing the house in case of damage from a covered peril, such as flood, earthquake, landslide, tornado, etc. The contents of the house may also be covered in case of damage or theft.
5. the health insurance (medical, dental, vision, etc. insurances) – will cover some or all health expenses of a policyholder.
6. the life insurance – will pay benefits to a designated beneficiary in case of a policyholder's death.
7. the disability income insurance – will replace all or portion of a policyholder's income in case of disability.
8. the life annuity – will make regular payments to a policyholder after the retirement until death.

Limits on Policy Benefits.

Definition. If an insurance does not reimburse the entire loss, the policyholder must cover part of the loss. This type of limit on policy benefits is called coinsurance. We study two types of coinsurances.

1. deductible – insurance will cover losses in excess of a stated amount. For example, a \$500 deductible on a car insurance means that all repairs that cost \$500 or less must be covered by the policyholder. If the cost exceeds \$500, the policyholder must pay \$500, and the insurance will pay the rest.
2. benefit limit – insurance will not cover losses beyond a stated upper bound.

Definition. Inflation is a persistent increase in the amount of money and credit in relation to the supply of goods and services which results in increase of prices and decline of purchasing power.

What is the role of an actuary?

1. determine the net and gross premiums of a policy.
2. determine the amount of assets the insurer should have on hand to assure that benefits can be paid as they arise.
3. project potential profit or loss of a new kind of policy.
4. assess potential difficulties of a new policy before they become significant.

Textbook: *Mathematical Statistics with Applications* by Wackerly, D.D., Mendenhall, W., and Sheaffer, R.L., Duxbury, 2008, 7th edition.

2.3 – 2.10, not 2.6 General Probability, Set Notation.

Definition. The sample space S is the set of all possible outcomes of a random experiment.
Definition. An event is a subset of S.
Notation. Events are denoted by A, B, C, F, G, A_1, etc. To list the elements in a set A write $A = \{a_1, a_2, \dots, a_n\}$.
Definition. The union $A \cup B$ of two events A and B contains all outcomes that are either in A or in B or in both.
Definition. The intersection $A \cap B$ of two events contains all outcomes that are in both, A and B.
Definition. Two events A and B are mutually exclusive or disjoint if their intersection is an empty set: $A \cap B = \emptyset$.
Definition. The complement A^c (or $\bar{A}$ or A' or $\sim A$ or A^C) of an event A contains all outcomes that are in S but not in A.
Definition. Draw Venn Diagram. Show union, intersection, mutually exclusive events, complement.
Definition. The probability of an event A, denoted $\mathbb{P}(A)$, is a number with the following two properties:
(1) $\mathbb{P}(A) \geq 0, \ \forall A$.
(2) $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ for any disjoint events A_i.
Useful Formulas. Draw Venn diagrams.

1. The additive law for two events: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
2. The additive law for three events: $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$.
3. The complement rule: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
4. If A_i, $i = 1, \dots, n$ are disjoint, then $\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbb{P}(A_i)$.
5. DeMorgan's laws: $\mathbb{P}((A \cap B)^c) = \mathbb{P}(A^c \cup B^c)$, $\mathbb{P}((A \cup B)^c) = \mathbb{P}(A^c \cap B^c)$.

Definition. Two events A and B are independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.
Definition. If events $A_1, A_2, \dots$ are independent then $\mathbb{P}(\cap A_i) = \prod \mathbb{P}(A_i)$.
Definition. The conditional probability of A given B is

$$\mathbb{P}(A|\,B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Useful Formula $\mathbb{P}(A \cap B) = \mathbb{P}(A|\,B)\mathbb{P}(B)$. This is the multiplicative law.
Definition. A set $\{A_1, A_2, \dots, A_n\}$ is a partition of the sample space S if (1) A_i are mutually exclusive, and (2) $\cup A_i = S$. Draw Venn Diagram.
The Bayes Theorem. Let $\{A_1, A_2, \dots, A_n\}$ be a partition of S. Given that an event B has occurred, the updated probability of A_i is

$$\mathbb{P}(A_i|\,B) = \frac{\mathbb{P}(B|\,A_i)\,\mathbb{P}(A_i)}{\sum_{j=1}^n \mathbb{P}(B|\,A_j)\,\mathbb{P}(A_j)}.$$

Definition. A combination of objects is an unordered arrangement of the objects.
Definition. The number of combinations of k objects chosen from n distinct objects is given by the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
Definition. The number of ways to separate n distinct objects into k groups of sizes $n_1, \dots, n_k$ where $n_1 + \dots + n_k = n$ is given by the formula

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\dots\binom{n_k}{n_k} = \frac{n!}{n_1! \dots n_k!}.$$

3.1 – 3.9 Discrete Probability Distributions.

Definition. A random variable is a function that assigns a real number to every outcome in the sample space.
Definition. A discrete random variable assumes a finite or a countably infinity number of values.
Definition. The probability distribution of a discrete random variable X is the list of the values with the respective probabilities $\mathbb{P}(X = x) = p(x)$. The function $p(x)$ is called the probability function. It has the properties:
(i) $p(x) \geq 0\ \forall x$, and (ii) $\sum_x p(x) = 1$.
Definition. The mean (or expected value or expectation or average) of a discrete random variable X is $\mathbb{E}(X) = \sum_x x\,p(x)$.
Useful Formulas.
1. $\mathbb{E}(aX + b) = a\,\mathbb{E}(X) + b$.

2. $\mathbb{E}(g(X)) = \sum_x g(x)\, p(x)$.
3. $\mathbb{E}(f(X_1) + g(X_2)) = \mathbb{E}(f(X_1)) + \mathbb{E}(g(X_2))$.

Definition. The 100α's percentile of a distribution of a random variable is the value x such that $\mathbb{P}(X \leq x) = \alpha$.

Definition. The first quantile Q_1 of a distribution of X satisfies $\mathbb{P}(X < Q_1) = .25$.

Definition. The median M of a distribution of X satisfies $\mathbb{P}(X < M) = 0.5 = \mathbb{P}(X > M)$.

Definition. The third quantile Q_3 of a distribution of X satisfies $\mathbb{P}(X < Q_3) = .75$.

Definition. The mode of a distribution is a local maximum.

Definition. The kth moment of a random variable X is $\mathbb{E}(X^k)$.

Definition. The variance of a random variable X is $\mathbb{V}ar(X) = \mathbb{E}\left(X - \mathbb{E}(X)\right)^2$.

Useful Formulas.
1. $\mathbb{V}ar(X) = \mathbb{E}(X)^2 - (\mathbb{E}(X))^2$.
2. $\mathbb{V}ar(aX + b) = a^2 \mathbb{V}ar(X)$.
3. If X_1 and X_2 are independent, then $\mathbb{E}(f(X_1)g(X_2)) = \mathbb{E}(f(X_1)\,\mathbb{E}(g(X_2))$.
4. If X_1 and X_2 are independent, then $\mathbb{V}ar(f(X_1) + g(X_2)) = \mathbb{V}ar(f(X_1)) + \mathbb{V}ar(g(X_2))$.

Definition. The standard deviation of a random variable X is $\sigma = \sqrt{\mathbb{V}ar(X)}$.

Definition. The interquantile range of a distribution is $Q_3 - Q_1$, the difference between the third and the first quantiles.

Definition. The moment generating function (m.g.f.) of a random variable X is $m(t) = m_X(t) = \mathbb{E}(e^{tX})$.

Useful Formulas.
1. $\mathbb{E}(X) = m'(0)$, $\mathbb{E}(X^2) = m''(0)$. In general, $\mathbb{E}(X^n) = m^{(n)}(0)$.
2. If $X_1, \cdots, X_n$ are independent and $X = \sum_{i=1}^n X_i$, then $m_X(t) = \prod_{i=1}^n m_{X_i}(t)$.

Certain Discrete Distributions.

Name	Notation	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	$\mathbb{V}ar(X)$	$m(t)$
Binomial	$X \sim Bi(n,p)$	$\binom{n}{x}p^x(1-p)^{n-x}$, $x = 0, \ldots, n$	np	$np(1-p)$	$(pe^t + 1 - p)^n$
Geometric	$Geom(p)$	$p(1-p)^{x-1}$, $x = 1, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Negative Binomial	$NB(p,r)$	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$ $x = r, \ldots$	$\frac{r}{p}$	$\frac{(1-p)r}{p^2}$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$
Hypergeometric	$HG(N,n,k)$	$\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$, $x = 0, \ldots, n$,	$n\frac{k}{N}$	$n\frac{k}{N}(1-\frac{k}{N})\frac{N-n}{N-1}$	no closed form
Poisson	$Poi(\lambda)$	$\frac{\lambda^x}{x!}e^{-\lambda}$ $x = 0, 1, \ldots$	λ	λ	$e^{\lambda(e^t - 1)}$

4.2 – 4.6, 4.9 Continuous Probability Distributions.

Definition. A continuous random variable assumes values in an interval.
Definition. The probability density function (p.d.f.) of a continuous random variable Y is a function $f(y)$ with the properties: (i) $f(y) \geq 0 \; \forall y$,
(ii)$\int_{-\infty}^{\infty} f(y)\,dy = 1$, and (iii) $\mathbb{P}(a \leq Y \leq b) = \int_a^b f(y)\,dy$.
Definition. The cumulative distribution function (c.d.f.) of a continuous random variable Y is $F(y) = \mathbb{P}(Y \leq y) = \int -\infty^y f(u)\,du$.
Useful Formulas.
1. $f(y) = F'(y)$.
2. $\mathbb{P}(a \leq Y \leq b) = F(b) - F(a)$.
Definition. The mean (or expected value or expectation or average) of a continuous random variable Y is $\mathbb{E}(Y) = \int_{-\infty}^{\infty} y\,f(y)\,dy$.
Useful Formulas.
1. $\mathbb{E}(aY + b) = a\,\mathbb{E}(Y) + b$.
2. $\mathbb{E}\left(g(Y)\right) = \int_y g(y)\,f(y)\,dy$.
3. $\mathbb{E}(Y_1 + Y_2) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2)$.
4. $\mathbb{V}ar(Y) = \mathbb{E}(Y)^2 - (\mathbb{E}(Y))^2$.
5. $\mathbb{V}ar(aY + b) = a^2\mathbb{V}ar(Y)$.
6. If Y_1 and Y_2 are independent, then $\mathbb{V}ar(Y_1 + Y_2) = \mathbb{V}ar(Y_1) + \mathbb{V}ar(Y_2)$.

Certain Continuous Distributions.

Uniform: $U \sim Unif(a,b)$, $f(u) = \begin{cases} \frac{1}{b-a}, & if a \leq u \leq b \\ 0, & ow \end{cases}$, $F(u) = \begin{cases} 0, & if u < a \\ \frac{u-a}{b-a}, & if a \leq u \leq b, \\ 1, & if u \geq b \end{cases}$
$\mathbb{E}(U) = \frac{a+b}{2}$, $\mathbb{V}ar(U) = \frac{(b-a)^2}{12}$, $m(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$.

Exponential: $X \sim Exp(\beta)$, $f(x) = \frac{1}{\beta}\,e^{-x/\beta}$, $x > 0$, $\beta > 0$,
$F(x) = \begin{cases} 0, & if x \leq 0 \\ 1 - e^{-x/\beta}, & if x > 0 \end{cases}$ $\mathbb{E}(X) = \beta$, $\mathbb{V}ar(X) = \beta^2$, $m(t) = \frac{1}{1-\beta\,t}$.
Useful Facts:
1. If occurrences have $Poisson(\lambda)$ distribution, then the interarrival times are $Exp(1/\lambda)$.
2. Memoryless property of exponential distribution: $\mathbb{P}(X > t + s|\, X > t) = \mathbb{P}(X > s)$.

Gamma: $X \sim Gamma(\alpha, \beta)$, $f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\,\beta^\alpha}\,e^{-x/\beta}$, $x > 0, \alpha > 0$, $\beta > 0$,
$\Gamma(\alpha) = \int_0^\infty x^\alpha\,e^{-x}\,dx$, $\mathbb{E}(X) = \alpha\,\beta$, $\mathbb{V}ar(X) = \alpha\,\beta^2$, $m(t) = \left(\frac{1}{1-\beta\,t}\right)^\alpha$.

Useful Facts:
1. $\Gamma(n) = (n-1)!$ where n is an integer, $n \geq 1$.
2. Gamma r.v. is the sum of α i.i.d. exponentials.

Normal: $X \sim N(\mu, \sigma^2)$, $f(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$, $\mathbb{E}(X) = \mu$, $\mathbb{V}ar(X) = \sigma^2$, $m(t) = Exp\left(\mu\, t + \frac{\sigma^2\, t^2}{2}\right)$.

Lognormal: If $\log X \sim N(\mu, \sigma^2)$, then X has a lognormal distribution. The density of lognormal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, \frac{1}{x}\, e^{-(\log x-\mu)^2/(2\sigma^2)},\ x > 0,\ \ \mathbb{E}(X^n) = e^{n\mu+n^2\sigma^2/2},\ \ m(t)\ \nexists.$$

Chi-squared: $X \sim \chi^2(p)$, $f(x) = \frac{1}{\Gamma(p/2)\, 2^{p/2}}\, x^{(p/2)-1}\, e^{-x/2}$, $x > 0$, $\mathbb{E}(X) = p$, $\mathbb{V}ar(X) = 2p$, $m(t) = \left(\frac{1}{1-2t}\right)^{p/2}$, $t < 1/2$.

Useful Facts:

1. If $Z_i \overset{i.i.d.}{\sim} N(0,1)$, $i = 1, \ldots, n$, and $X = \sum_{i=1}^n Z_i^2 \sim \chi^2(n)$ has the chi-squared distribution with n degrees of freedom.
2. $\chi^2(n)$ is $Gamma(n/2, 1/2)$.

Beta: $X \sim Beta(\alpha,\, \beta)$, $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}ar(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, $m(t)$ doesn't look good.

Pareto: $X \sim Pareto(\alpha, \beta)$, $f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}$, $x > \alpha$, $\alpha > 0$, $\beta > 0$, $\mathbb{E}(X) = \frac{\beta\alpha}{\beta-1}$, $\mathbb{V}ar(X) = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}$, $\beta > 2$, $m(t)\ \nexists$.

Weibull: $X \sim Weibull(\gamma, \beta)$, $f(x) = \frac{\gamma}{\beta}\, x^{\gamma-1}\, e^{-x^\gamma/\beta}$, $x > 0, \gamma > 0, \beta > 0$, $\mathbb{E}(X^n) = \beta^{n/\gamma}\Gamma(1 + n/\gamma)$, $m(t)$ doesn't look good.

Useful Fact: If $X \sim Exp(\beta)$, then $X^{1/\gamma} \sim Weibull(\gamma, \beta)$.

6.4 Transformations.

Definition. Let X be a continuous random variable with p.d.f. f_X and c.d.f. F_X, and let g be a function. Define a transformation $Y = g(X)$. Then the c.d.f of Y is $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y) = \int_{x:\, g(x)\le y} f_X(x)\, dx$, and the p.d.f. of Y is $f_Y(y) = F_Y'(y)$.

In the special case when g is strictly increasing, $F_Y(y) = \mathbb{P}\left(X \le g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right)$, and $f_Y(y) = F_X'\left(g^{-1}(y)\right) \frac{dg^{-1}(y)}{dy}$. When g is strictly decreasing, $F_Y(y) = \mathbb{P}\left(X \ge g^{-1}(y)\right) = 1 - F_X\left(g^{-1}(y)\right)$ and $f_Y(y) = -F_X'\left(g^{-1}(y)\right) \frac{dg^{-1}(y)}{dy}$.

In general, if g is strictly increasing or decreasing, $f_Y(y) = F_X'\left(g^{-1}(y)\right) \left|\frac{dg^{-1}(y)}{dy}\right|$.

Order Statistics.

Definition. Let $X_1, \ldots, X_n$ be i.i.d. random variables with c.d.f. F and p.d.f. f. Suppose n realizations of these variables are observed. The observations in increasing order $X_{(1)}, \ldots, X_{(n)}$ are called order statistics. The p.d.f. of the ith order statistic is

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!}\, [F(x)]^{i-1}\, f(x)\, [1 - F(x)]^{n-i}\,.$$

Interesting special cases are

1. $F_{X_{(n)}}(x) = \mathbb{P}\left(X_{max} \le x\right) = \mathbb{P}(X_1 \le x, \ldots, X_n \le x) = \left[F(x)\right]^n,$

$$f_{X_{(n)}}(x) = n\left[F(x)\right]^{n-1}\, f(x).$$

2. $1 - F_{X_{(1)}}(x) = \mathbb{P}\left(X_{min} \ge x\right) = \mathbb{P}(X_1 \ge x, \ldots, X_n \ge x) = \left[1 - F(x)\right]^n,$

$$f_{X_{(1)}}(x) = n\left[1 - F(x)\right]^{n-1}\, f(x).$$

5.2 – 5.8 Multivariate Probability Distributions.

Definition. Let X_1 and X_2 be two discrete random variables. The joint probability function of X_1 and X_2 is $p\,(x_1, x_2) = \mathbb{P}(X_1 = x_1,\, X_2 = x_2)$.
Definition. Let Y_1 and Y_2 be two continuous random variables. The joint c.d.f is $F(y_1,\, y_2) = \mathbb{P}(Y_1 \le y_1,\, Y_2 \le y_2)$. The joint density is

$$f(y_1,\, y_2) = \frac{\partial^2 F(y_1,\, y_2)}{\partial y_1 \partial y_2},$$

or, equivalently, $F(y_1,\, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(u,\, v)\, dv\, du$.
Definition. The marginal probability distribution of X_1 is $p_1(x_1) = \sum_{x_2} p\,(x_1,\, x_2)$, of X_2 is $p_1(x_1) = \sum_{x_1} p\,(x_1,\, x_2)$.
Definition. The marginal density of Y_1 is $f_1(y_1) = \int_{-\infty}^{\infty} f(y_1,\, y_2)\, dy_2$, of Y_2 is $f_2(y_2) = \int_{-\infty}^{\infty} f(y_1,\, y_2)\, dy_1$.
Definition. The conditional probability function of X_1 given that $X_2 = x_2$ is

$$p(x_1 |\, x_2) = \frac{p\,(x_1,\, x_2)}{p_2(x_2)}.$$

Definition. The conditional density of Y_1 given that $Y_2 = y_2$ is

$$f(y_1 |\, y_2) = \frac{f(y_1,\, y_2)}{f_2(y_2)}.$$

Useful Formula.

$$\mathbb{P}(Y_1 \le y_1 |\, Y_2 = y_2) = \frac{\int_{-\infty}^{y_1} f(u,\, y_2)\, du}{\int_{-\infty}^{\infty} f(u,\, y_2)\, du}.$$

Definition. The joint m.g.f. of X and Y is $m(t, s) = e^{t\,X + s\,Y}$.

Definition. The covariance between two random variables X and Y is

$$\mathbb{C}ov(X,\, Y) = \mathbb{E}\left[(X - \mathbb{E}(X))\,(Y - \mathbb{E}(Y))\right] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

Properties.
1. $\mathbb{C}ov(X\,Y) = \mathbb{C}ov(Y,\, X)$.
2. $\mathbb{C}ov(aX + b,\, cY + d) = ac\mathbb{C}ov(X,\, Y)$.

3. $\mathbb{C}ov(X+Y,\, Z) = \mathbb{C}ov(X,\, Z) + \mathbb{C}ov(Y,\, Z)$.
4. If X and Y are independent, then $\mathbb{C}ov(X,\, Y) = 0$, that is, X and Y are uncorrelated. The converse is not true.
5. $\mathbb{V}ar(aX + bY + c) = a^2\mathbb{V}ar(X) + 2ab\mathbb{C}ov(X,\, Y) + b^2\mathbb{V}ar(Y)$.

Definition.; The correlation coefficient between X and Y is

$$\rho_{XY} = \frac{\mathbb{C}ov(X,\, Y)}{\sqrt{\mathbb{V}ar(X)}\,\sqrt{\mathbb{V}ar(Y)}}.$$

Properties.
1. $-1 \le \rho \le 1$.
2. ρ measures the direction and strength of linear relationship between X and Y. If X and Y are independent, $\rho = 0$. If $Y = aX + b$, then $\rho = sign(a) = \pm 1$.

7.3 The Central Limit Theorem (CLT).

Definition. The sample mean of random variables $W_1, \ldots, W_n$ is the arithmetic average

$$\bar{W} = \frac{W_1 + \cdots + W_n}{n}.$$

Theorem (The Central Limit Theorem). Let $W_1, \ldots, W_n$ be i.i.d. random variables with mean μ and standard deviation σ. Then for large n, $\bar{W}$ is approximately normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$. Equivalently, $W_1 + \cdots + W_n$ is approximately normally distributed with mean $n\mu$ and standard deviation $\sqrt{n}\,\sigma$.

Textbook: *Actuarial Mathematics* by Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J., Society of Actuaries, 1997, 2nd edition.

1.2 Utility Theory.
Lets play the following game. We flip a fair coin three times. If we see exactly one head, I pay you \$10, otherwise you pay me \$7. Do you want to play the game?
SOLUTION: The sample space of the random experiment is

$$S = \{HHH,\, HHT,\, HTH,\, THH,\, HTT,\, THT,\, TTH,\, TTT\}.$$

Denote by W your gain in the game. Then

$$\mathbb{P}(W = 10) = \mathbb{P}(\{HTT,\, THT,\, TTH\}) = 3/8,$$

and $\mathbb{P}(W = -7) = 1 - 3/8 = 5/8$. Your expected gain in the game is $\mathbb{E}(W) = (10)(3/8) + (-7)(5/8) = -5/8 = -.625$, that is, you expect to lose 62.5 cents. So, most likely you don't want to play the game.
Which game would you be willing to play? Only if $\mathbb{E}(W) \ge 0$. But if $\mathbb{E}(W) > 0$, I wouldn't want to play with you. So, the only way we can agree

to play the game is if $\mathbb{E}(W) = 0$. That is, we both expect to win nothing (a fair game). For example, if I pay you \$10 with probability 3/8 and you pay me \$6 with probability 5/8.

Definition. When making a decision in a situation that involves randomness, one approach is to replace the distribution of possible outcomes by the expected value of the outcomes. This approach is called the expected value principle.

Definition. In economics, the expected value of random prospects with monetary payments is called the fair value (or actuarial value) of the prospect.

We might expect a similar principle to be applicable in insurance business. It turns out that it is not always so. Consider the following situation. Suppose you own $w =$ \$100 which you might lose with probability $p = 0.01$. If you lose your wealth, an insurer offers to reimburse you in full (called a complete coverage or a complete insurance). How much would you be willing to pay for the prospect?

SOLUTION: Notice that if not insured you expect to lose $wp = (100)(.01) =$ \$1. Suppose you are willing to pay \$$x$ for the insurance. The insurer's gain then equals x with probability $1-p = .99$ and $x-100 = x-w$ with probability $p = .01$. Thus, the insurer's expected gain is $x(1-p)+(x-w)p = x-wp$. If it were to be a fair game, $x - wp = 0$ or $x = wp$, that is, you should be willing to pay the amount of your expected loss $wp =$ \$1. Would you be willing to pay more than \$1? Most likely not, if your wealth is just \$100. But suppose your wealth is one million dollars. Then you should be willing to pay the insurer the amount of the expected loss of $(1,000,000)(.01) =$ \$10,000. Would you be willing to pay more than that? Most likely yes, because if not insured, there is a chance of a catastrophic loss. But how much more are you willing to pay? It depends on a person.

Definition. The value (or utility) that a particular decision-maker attaches to wealth of amount w, can be specified in the form of a function $u(w)$, called a utility function.

How to determine values of an individual's utility function?

Example. Suppose a decision-maker has wealth $w =$ \$20,000. So, his utility function is defined on interval $[0,\ 20,000]$. We chose arbitrarily the endpoints of his utility function, for example, $u(0) = -1$ and $u(20,000) = 0$. To determine the values of the utility function in intermediate points, proceed as follows. Ask the decision-maker the following question: Suppose you can lose your \$20,000 with probability 0.5. How much would you be willing to pay an insurer for a complete insurance against the loss? That is, define the maximum amount G such that $u(20,000 - G) = (0.5)\,u(20,000) + (0.5)\,u(0) = (0.5)(0) + (0.5)(-1) = -0.5$. Here the left-hand side represents the utility of the insured certain amount \$$20,000 - G$, while the right-hand side is

the expected utility of the uninsured wealth. Suppose the decision-maker defines $G = \$12,000$. Therefore, $u(8,000) = -0.5$. Notice that the decision maker is willing to pay for the insurance more than the expected loss $(0.5)(20,000) + (0.5)(0) = 10,000$.
To determine the other values of the utility function, ask the question: What is the maximum amount you would pay for a complete insurance against a situation that could leave you with wealth w_2 with probability p, or at reduced wealth w_1 with probability $1-p$? That is, we ask the decision maker to specify G such that $u(w_2 - G) = p\,u(w_2) + (1-p)\,u(w_1)$. For example, $G = 7,500$ in the situation $u(20,000 - G) = (0.5)\,u(20,000) + (0.5)\,u(8,000) = (0.5)(0) + (0.5)(-0.5) = -.25$. This defines $u(12,500) = -0.25$. Notice that G again exceeds the expected loss of $(0.5)(0) + (0.5)(12,000) = 6,000$. Picture.

The main theorem of the utility theory states that a decision-maker prefers the distribution of X to the distribution of Y, if $\mathbb{E}(u(X)) > \mathbb{E}(u(Y))$ and is indifferent between the two distributions, if $\mathbb{E}(u(X)) = \mathbb{E}(u(Y))$.

1.3. Insurance and Utility.

We apply the utility theory to the decision problems faced by a property owner. Suppose the random loss X to his property has a known distribution. The owner will be indifferent between paying a fixed amount G to an insurer, or assuming the risk himself. That is, $u(w - G) = \mathbb{E}(u(w - X))$.
Remember that to make a profit, an insurer must charge a premium that exceeds the expected loss, that is, it should be true that $G > \mathbb{E}(X) = \mu$. Which utility functions satisfy this property?

Proposition. If $u'(w) > 0$ and $u''(w) < 0$, then $G > \mu$.
PROOF: We make use of Jensen's inequality which states that if $u''(w) < 0$, then $\mathbb{E}(u(X)) \leq u(\mathbb{E}(X))$. The exact equality holds iff $X = \mu$.
By this inequality, $u(w-G) = \mathbb{E}(u(w-X)) \leq u(\mathbb{E}(w-X)) = u(w-\mu)$. Now, since $u'(w) > 0$, u is an increasing function, and therefore, $w - G \leq w - \mu$ or $\mu \leq G$ with $\mu < G$ unless X is a constant. □
Definition. A decision-maker with utility function $u(w)$ is <u>risk averse</u> if $u''(w) < 0$, and a <u>risk lover</u> if $u''(w) > 0$.
Remark. According to the proposition, for risk averse people $G \geq \mu$ and, so, they are able to get a complete insurance. It can be shown that for risk lovers $G \leq \mu$ and, so, they won't be insured.

Three functions are commonly used to model utility functions.
MODEL 1. An <u>exponential</u> utility function is of the form $u(w) = -e^{-\alpha w}$, where $w > 0$ and $\alpha > 0$ is a constant. It has the following properties: (1) $u'(w) = \alpha\, e^{-\alpha w} > 0$, (2) $u''(w) = -\alpha^2\, e^{-\alpha w} < 0$, (3) G doesn't depend on w. To see that, write $-e^{-\alpha\,(w-G)} = \mathbb{E}\big[-\,e^{-\alpha\,(w-X)}\big] = -\,e^{-\alpha\, w}\, M_X(\alpha)$ where M_X is the m.g.f. of X. From here, $G = \ln M_X(\alpha)/\alpha$.

Example 1.3.1. A decision-maker has an exponential utility function $u(w) = -e^{-5w}$. Suppose he faces two economic prospects with outcomes $X \sim N(5, 2)$ and $Y \sim N(6, 2.5)$, respectively. Which prospect should be preferred?
SOLUTION: $\mathbb{E}(u(X)) = -M_X(-5) = -e^{(5)(-5)+(2)(-5)^2/2} = -1$, and $\mathbb{E}(u(Y)) = -e^{(6)(-5)+(2.5)(-5)^2/2} = -e^{1.25} = -3.49 < -1 = \mathbb{E}(u(X))$, so the distribution of X should be preferred to the distribution of Y.

MODEL 2. A fractional power utility function is of the form $u(w) = w^\gamma$, where $w > 0$ and $0 < \gamma < 1$ is a constant. Check that $u'(w) > 0$ and $u''(w) < 0$.

Example 1.3.2. Suppose $u(w) = \sqrt{w}$, $w = 10$ and $X \sim Unif(0, 10)$. Find G.
SOLUTION: $\sqrt{10 - G} = \mathbb{E}(\sqrt{10 - X}) = \int_0^{10} \sqrt{10 - X}\, \frac{1}{10}\, dx = \frac{2}{3}\sqrt{10}$, so $10 - G = 40/9$ and $G = 10 - 40/9 = 5.56$.

MODEL 3. A quadratic utility function is of the form $u(w) = w - \alpha\, w^2$, where $w < 1/(2\alpha)$, and $\alpha > 0$ is a constant. Check that $u'(w) > 0$ and $u''(w) < 0$.

Example 1.3.3. Suppose $u(w) = w - 0.01\, w^2$, $w < 50$. Also suppose that with probability $p = 0.5$ the decision maker will retain wealth of amount $w = 20$ and with probability $1 - p = 0.5$ will suffer a financial loss of amount $c = 10$. Find G.
SOLUTION: Since $u(w - G) = pu(w) + (1 - p)u(w - c)$, we have $20 - G - 0.01(20 - G)^2 = (0.5)(20 - 0.01(20)^2) + (0.5)(10 - 0.01(10)^2)$. Check that G solves the quadratic equation $0.01\, G^2 + 0.6\, G - 3.5 = 0$. Thus, $G = 5.36$.

Example 1.3.4 A property will not be damaged with probability 0.75. A positive loss has $Exp(100)$ distribution. The owner of the property has a utility function $u(w) = -e^{-0.005w}$. Find $\mathbb{E}(X)$ and G.

SOLUTION: Denote the loss by X. Then X has a mixed distribution. X has a mass 0.75 at zero and is $Exp(100)$ with probability 0.25. The expected loss $\mathbb{E}(X) = (0.75)(0) + (0.25)(100) = 25$. To find G, write

$$u(w-G) = \mathbb{E}(u(w-X)) = (0.75)u(w) + (0.25)\int_0^\infty u(w-x)(0.01)\, e^{-0.01x}\, dx,$$

$$-e^{-0.005(w-G)} = -(0.75)\, e^{-0.005w} - (0.25)\int_0^\infty e^{-0.005(w-x)}(0.01)\, e^{-0.01x}\, dx,$$

$$e^{0.005G} = 0.75 + (0.25)(2) = 1.25, \qquad G = 200 \ln(1.25) = 44.63.$$

The property owner is willing to pay up to $G - \mathbb{E}(X) = 44.63 - 25 = 19.63$ in excess of the expected loss to purchase the complete insurance.

1.5. Optimal Insurance.

Theorem 1.5.1. Suppose a decision maker (1) has wealth w, (2) has a utility function $u(w)$ such that $u'(w) > 0$ and $u''(w) < 0$ (a risk averse), (3) faces a random loss X, and (4) is willing to spend amount P purchasing an insurance. Suppose also that the insurance market offers for a payment P all feasible insurance contracts of the form $I(x)$ where $0 \leq I(x) \leq x$ (avoiding an incentive to incur the loss) with expected payoff $\mathbb{E}(I(X)) = \beta$. Then to maximize the expected utility, the decision maker should choose an insurance policy

$$I_{d^*}(x) = \begin{cases} 0 & \text{if } x < d^*, \\ x - d^* & \text{if } x \geq d^*. \end{cases}$$

where d^* is the unique solution of $\mathbb{E}(I_d(X)) = \int_d^\infty (x-d) f(x)\,dx = \beta$.
Definition. A feasible insurance contract of the form

$$I_d(x) = \begin{cases} 0 & \text{if } x < d \\ x - d & \text{if } x \geq d \end{cases}$$

pays losses above the deductible amount d. This type of contract is called stop-loss or excess-of-loss insurance.

Example. Assume $w = 100$ and $X \sim Unif(0, 100)$. Then by the theorem d^* solves

$$\beta = \int_d^{100} (x-d)\frac{1}{100}\,dx = \frac{1}{200}\,d^2 - d + 50.$$

That is, d^* solves the quadratic equation $d^2 - 200d + 10,000 - 200\beta = 0$, or $d^* = 100 - \sqrt{200\beta}$. For example,

Expected payoff β	Deductible d^*
0	100
10	55.28
20	36.75
30	22.54
40	10.56
45	5.13
50	0

Note that there is no need to specify $u(w)$ and P.

2.2. Models for Individual Claim Random Variables.

We will consider three individual risk models for short time periods. These models do not take into account the inflation of money.

MODEL 1. In a one-year term life insurance the insurer agrees to pay an amount b if the insured dies within a year of policy issue and to pay nothing if the insured survives the year. The probability of a claim during the year is denoted by q. The claim random variable X has distribution

$$\mathbb{P}(X = x) = \begin{cases} 1-q & x = 0 \\ q & x = b \\ 0 & \textit{otherwise}. \end{cases}$$

Notice that $X = bI$ where $I \sim Bernoulli(q)$ indicates whether a death has occurred and, therefore, is called an indicator. Thus, the expected claim $\mathbb{E}(X) = bq$ and $\mathbb{V}ar(X) = b^2 q(1-q)$.

MODEL 2. Consider the above model $X = IB$ where the claim amount B varies. Suppose if death is accidental, the benefit amount is $B = \$50,000$, otherwise, $B = \$25,000$. Suppose also that the probability of an accidental death within the year is .0005, and the probability of a nonaccidental death is .002. That is,

$$\mathbb{P}(I = 1,\ B = 50,000) = .0005, \quad \mathbb{P}(I = 1,\ B = 25,000) = .002.$$

Hence, $\mathbb{P}(I = 1) = .0005 + .002 = .0025$, and $\mathbb{P}(I = 0) = .9975$. Therefore, the distribution of X is $\mathbb{P}(X = 0) = .9975$, $\mathbb{P}(X = 25,000) = .002$, $\mathbb{P}(X = 50,000) = .0005$. The expectation $\mathbb{E}(X) = \$75$. Also, the conditional distribution of B, given $I = 1$, is

$$\mathbb{P}(B = 25,000 | I = 1) = \frac{\mathbb{P}(B = 25,000,\ I = 1)}{\mathbb{P}(I = 1)} = \frac{.002}{.0025} = .8,$$

and

$$\mathbb{P}(B = 50,000 | I = 1) = \frac{\mathbb{P}(B = 50,000,\ I = 1)}{\mathbb{P}(I = 1)} = \frac{.0005}{.0025} = .2.$$

This means that 20% of all payoffs are for accidental deaths, and 80% are for nonaccidental. The expected payoff is (25,000)(.8)+(50,000)(.2)=\$30,000.

MODEL 3. Consider an automobile collision coverage above a \$250 deductible up to a maximum claim of \$2,000. Assume that for an individual the probability of one claim in a period is .15, and the probability of more than one claim is zero, that is, $\mathbb{P}(I = 1) = .15$, and $\mathbb{P}(I = 0) = .85$. Assume also that

$$\mathbb{P}(B \le x | I = 1) = \begin{cases} 0 & \text{if } x \le 0 \\ (.9)\left[1 - (1 - x/2,000)^2\right] & \text{if } 0 < x < 2,000 \\ 1 & \text{if } x \ge 2,000. \end{cases}$$

Notice that B has a mixed distribution with a mass at 2,000. The distribution of the claim random variable X is

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(BI \leq x |\, I=0)\mathbb{P}(I=0) + \mathbb{P}(BI \leq x |\, I=1)\mathbb{P}(I=1)$$

$$= \begin{cases} 0 & \text{if } x \leq 0 \\ (1)(.85) + (.15)(.9)\left[1 - (1 - x/2,000)^2\right] & \text{if } 0 < x < 2,000 \\ 1 & \text{if } x \geq 2,000. \end{cases}$$

The density of X is $f_X(x) = F_X'(x) = .000135\,(1 - x/2,000)$ if $0 < x < 2,000$, $\mathbb{P}(X=0) = .85$, and $\mathbb{P}(X = 2,000) = .015$. The kth moment of X is $\mathbb{E}(X^k) = (2,000)^k(.015) + \int_0^{2,000} x^k f_X(x)\,dx$. Check that $\mathbb{E}(X) = 120$ and $\mathbb{V}ar(X) = 135,600$.

2.3. Sums of Independent Random Variables.

Proposition. (1) Suppose X and Y are two independent discrete random variables, and let $S = X + Y$. Then the distribution of S is $F_S(s) = \mathbb{P}(S \leq s) = \mathbb{P}(X + Y \leq s) = \sum_{y=0}^{s} \mathbb{P}(X + Y \leq s |\, Y = y)(Y = y) = \sum_{y=0}^{s} \mathbb{P}(X \leq s - y)\mathbb{P}(Y = y)$. Also, $\mathbb{P}(S = s) = \sum_{y=0}^{s} \mathbb{P}(X = s - y)\mathbb{P}(Y = y)$.
(2) Suppose X and Y are two independent continuous random variables. Then $F_S(s) = \int_0^s F_X(s-y) f_Y(y)\,dy$ and $f_S(s) = \int_0^s f_X(y - s)\, f_Y(y)\,dy$.

Definition. The <u>convolution</u> of $F_X(x)$ and $F_Y(y)$ is

$$F_X * F_Y = \int_0^s F_X(s-y)\, f_Y(y)\, dy.$$

Example 2.3.2. Let $X \sim Unif(0, 2)$ be independent of $Y \sim Unif(0, 3)$. Find the c.d.f. of $S = X + Y$.
SOLUTION:

$$F_S(s) = \begin{cases} 0 & \text{if } s < 0 \\ \int_0^s \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s^2}{12} & \text{if } 0 \leq s < 2 \\ \int_0^{s-2} 1\frac{1}{3}\,dy + \int_{s-2}^{s} \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s-1}{3} & \text{if } 2 \leq s < 3 \\ \int_0^{s-2} 1\frac{1}{3}\,dy + \int_{s-2}^{3} \frac{s-y}{2}\frac{1}{3}\,dy = 1 - \frac{(5-s)^2}{12} & \text{if } 3 \leq s < 5 \\ 1 & \text{if } s \geq 5. \end{cases}$$

Sometimes one can use the <u>method of moment generating functions</u> to find the distribution of S.
Proposition. Suppose $X_i,\ i = 1, \ldots, n$ are independent random variables with m.g.f.'s $M_{X_i}(t)$. Then the m.g.f. of $S = X_1 + \cdots + X_n$ is $M_S(t) = \prod_{i=1}^n M_{X_i}(t)$.
PROOF: $M_S(t) = \mathbb{E}\left(e^{tS}\right) = \mathbb{E}\left(e^{t(X_1 + \cdots + X_n)}\right) = \{\text{independence}\} = \prod_{i=1}^n \mathbb{E}\left(e^{tX_i}\right) = \prod_{i=1}^n M_{X_i}(t)$. □

Example. (1) $X_i \overset{i.i.d.}{\sim} Bernoulli(p),\ i = 1, \ldots, n$. Then $M_X(t) = pe^t + 1 - p$,

and $M_S(t) = (pe^t + 1 - p)^n$, and therefore $S \sim Bi(n, p)$.
(2) Show that the sum of r independent $Geom(p)$ random variables is $NB(r, p)$.
(3) Show that the sum of α independent $Exp(\beta)$ random variables is $Gamma(\alpha, \beta)$.
(4) Show that the sum of n independent $Poi(\lambda)$ random variables is $Poi(n\lambda)$.
(5) Show that the sum of n independent $N(\mu, \sigma^2)$ random variables is $N(n\mu, n\sigma^2)$.

2.5. Applications of the Central Limit Theorem to Insurance.

Example 2.5.1. The table below gives the number of insured n_k, the benefit amount b_k, and the probability of claim q_k where $k = 1, \ldots, 4$.

k	n_k	b_k	q_k	$b_k q_k$	$b_k^2 q_k(1 - q_k)$
1	500	1	.02	.02	.0196
2	500	2	.02	.04	.0784
3	300	1	.10	.10	.0900
4	500	2	.10	.20	.3600

The life insurance company wants to collect the amount equal to 95th percentile of the distribution of total claims. The share of the jth insured is $(1 + \theta)\mathbb{E}(X_j)$. The amount $\theta\mathbb{E}(X_j)$ is called the security loading and θ is called the relative security loading. This way the company protects itself from the loss of funds due to excess claims. Find θ.
SOLUTION: We want to find θ such that $\mathbb{P}(S \le (1 + \theta)\mathbb{E}(S)) = .95$. We use the CLT to obtain

$$\mathbb{P}\left(Z \le \frac{\theta\mathbb{E}(S)}{\sqrt{\mathbb{V}ar(S)}}\right) = .95,$$

$$\frac{\theta\mathbb{E}(S)}{\sqrt{\mathbb{V}ar(S)}} = 1.645.$$

$\mathbb{E}(S) = \sum_{k}, n_k\, b_k\, q_k = 160$, $\mathbb{V}ar(S) = \sum_k n_k\, b_k^2\, q_k\,(1 - q_k) = 256$. Therefore, $\theta = .1645$.

Example 2.5.3. A life insurance benefits are $q_1 = \cdots = q_5 = .02$ and

Benefit amount	Number insured
10,000	8,000
20,000	3,500
30,000	2,500
50,000	1,500
100,000	500

The retention limit is the amount below which this company (the ceding company) will retain the insurance and above which it will purchase reinsurance coverage from another (the reinsuring) company. Suppose the insurance company sets the retention limit at 20,000. Suppose also that reinsurance is available at a cost of .025 per unit of coverage. It is assumed that the model is a closed model, that is, the number of insured units is known and doesn't change during the covered period. Otherwise, the model allows migration in and out of the insurance system and is called an open model. Find the probability that the company's retained claims plus cost of reinsurance exceeds 8,250,000.

SOLUTION: Lets work in units of \$10,000. Let S be the amount of retained claims paid. The portfolio of retained business is

k	b_k	n_k
1	1	8,000
2	2	8,000

$\mathbb{E}(S) = 480$, $\mathbb{V}ar(S) = 784$. The total coverage in the plan is $(8,000)(1) + (3,500)(2) + \cdots + (500)(10) = 35,000$, and the retained coverage is $(8,000)(1) + (8,000)(2) = 24,000$. The difference $35,000 - 24,000 = 11,000$ is the reinsured amount. The reinsurance cost is $(11,000)(.025) = 275$. Thus, the retained claims plus the reinsurance cost is $S + 275$. We need to compute $\mathbb{P}(S + 275 > 825) = \mathbb{P}(S > 550) = \mathbb{P}\left(Z > \frac{550-480}{\sqrt{784}}\right) = \mathbb{P}(Z > 2.5) = .0062$.

12.1. Collective Risk Models for a Single Period. Introduction.

Definition. The collective risk model is the model of the aggregate claim amount generated by a portfolio of policies. Denote by N the number of claims generated by a portfolio of policies in a given time period. Let X_i be the amount of the ith claim (severity of ith claim). Then $S = X_1 + \cdots + X_N$ is the aggregate claim amount. The variables N, $X_1, \ldots, X_N$ are random variables such that (1) X_i are identically distributed and (2) N, $X_1, \ldots, X_n$ are independent.

12.2. The Distribution of Aggregate Claims.

Notation. Denote by $p_k = \mathbb{E}(X^k)$ the kth moment of the i.i.d. X_i's. Let $M_X(t) = \mathbb{E}\left[e^{tX}\right]$ be the m.g.f. of X_i. Also, let $M_N(t)$ and $M_S(t)$ denote the m.g.f.'s of N and S, respectively.

Proposition. (1) $\mathbb{E}(S) = \mathbb{E}[\mathbb{E}(S|\, N)] = \mathbb{E}[p_1 N] = p_1 \mathbb{E}(N)$.

(2) $\mathbb{V}ar(S) = \mathbb{V}ar[\mathbb{E}(S|\, N)] + \mathbb{E}[\mathbb{V}ar(S|\, N)] = \mathbb{V}ar[p_1 N] + \mathbb{E}[(p_2 - p_1^2)\, N] = p_1^2 \mathbb{V}ar(N) + (p_2 - p_1^2)\, \mathbb{E}(N)$.

(3) $M_S(t) = \mathbb{E}[\mathbb{E}(e^{tS}|\, N)] = \mathbb{E}[M_X(t)^N] = \mathbb{E}\left[e^{N \ln M_X(t)}\right] = M_N(\ln M_X(t))$.

Examples 12.2.1 and 12.2.3. Let $N \sim Geom(p)$ and $X_i \sim Exp(1)$. Find the distribution of S.

SOLUTION:

$$M_N(t) = \frac{p}{1-(1-p)e^t} \quad \text{and} \quad M_X(t) = \frac{1}{(1-t)}.$$

Thus,

$$M_S(t) = \frac{p}{1-(1-p)/(1-t)} = p + (1-p)\frac{p}{p-t},$$

which is a weighted average of two m.g.f.'s (0 and $Exp(p)$) with weights p and $1-p$, respectively. Therefore, S has a mixed distribution with mass p at zero and $Exp(p)$ distribution for $x > 0$. That is,

$$F_S(x) = \begin{cases} 0 & \text{if } x < 0 \\ (p)(1) + (1-p)(1-e^{-px}) = 1-(1-p)e^{-px} & \text{if } x \geq 0. \end{cases}$$

Picture.

12.3.1. The Distribution of N.

Three distributions are commonly used.
1. $N \sim Poi(\lambda)$. The distribution of S is called a compound Poisson distribution.

$$\mathbb{E}(S) = p_1\,\lambda,\ \ \mathbb{V}ar(S) = p_1^2\,\lambda + (p_2 - p_1^2)\,\lambda = p_2\,\lambda,\ \text{and}\ M_S(t) = e^{\lambda[M_X(t)-1]}.$$

Recall that if $N \sim Poi(\lambda)$, then $\mathbb{E}(N) = \mathbb{V}ar(N) = \lambda$. If the variance of N is larger than the mean, the Poisson distribution is not appropriate. Then a negative binomial distribution is used.

2. $N \sim NB(r,p)$. The distribution of S is called a compound negative binomial distribution.

$$\mathbb{E}(S) = p_1\,\frac{r(1-p)}{p},\ \ \mathbb{V}ar(S) = p_1^2\,\frac{r(1-p)^2}{p^2} + p_2\,\frac{r(1-p)}{p},\ \text{and}$$

$$M_S(t) = \left[\frac{p}{1-(1-p)\,M_X(t)}\right]^r.$$

Notice that indeed if $N \sim NB$, then $\mathbb{V}ar(N) > \mathbb{E}(N)$.

3. N has a conditional (or mixed) Poisson distribution: $N|\,\Lambda \sim Poi(\Lambda)$ where Λ is a random variable with p.d.f. $u(\lambda)$. We have

$$\mathbb{P}(N=n) = \int_0^\infty \mathbb{P}(N=n|\,\Lambda=\lambda)\,u(\lambda)\,d\lambda = \int_0^\infty \frac{\lambda^n}{n!}\,e^{-\lambda}\,u(\lambda)\,d\lambda,$$

$$\mathbb{E}(N) = \mathbb{E}[\mathbb{E}(N|\,\Lambda)] = \mathbb{E}(\Lambda),\ \ \mathbb{V}ar(N) = \mathbb{E}(\Lambda) + \mathbb{V}ar(\Lambda),\ \ M_N(t) = M_\Lambda(e^t - 1).$$

Notice that $\mathbb{E}(N) < \mathbb{V}ar(N)$, as in the case of the negative binomial distribution. In fact, the negative binomial distribution can be derived in this fashion.

Example 12.3.1. Let $\Lambda \sim Gamma(\alpha,\,\beta)$. Show that

$$N|\,\Lambda \sim NB\left(r=\alpha,\,p=\frac{\beta}{1+\beta}\right).$$

SOLUTION: $M_\Lambda(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$. Thus,

$$M_N(t) = M_\Lambda(e^t-1) = \left(\frac{\beta}{\beta-(e^t-1)}\right)^\alpha = \left[\frac{\beta/(1+\beta)}{1-[1-\beta/(1+\beta)]\,e^t}\right]^\alpha. \quad \square$$

12.5. Approximations to the Distribution of Aggregate Claims.

The following versions of the central limit theorem hold for compound Poisson and compound negative binomial distributions.

Theorem 12.5.1. (1) If S has a compound Poisson distribution, then for large λ,

$$Z = \frac{S - \lambda\, p_1}{\sqrt{\lambda\, p_2}}$$

has approximately $N(0,1)$ distribution.

(2) If S has a compound negative binomial distribution, then for large r,

$$Z = \frac{S - rp_1 \frac{1-p}{p}}{\sqrt{rp_2 \frac{1-p}{p} + rp_1^2 \frac{(1-p)^2}{p^2}}}$$

has approximately $N(0,1)$ distribution.

The normal distribution doesn't give a good approximation, since the it is a symmetric distribution, but the distribution of aggregate claims is skewed to the right (has a long right tail). A translated gamma distribution gives a better approximation.

Definition. Let $G(x,\, \alpha,\, \beta)$ denote the c.d.f of a $Gamma(\alpha,\, \beta)$ distribution. Then $H(x,\, \alpha,\, \beta,\, x_0) = G(x - x_0,\, \alpha,\, \beta)$ is the c.d.f. of a translated gamma distribution. Picture.

The parameters α, β, x_0 are found by equating the first moments

$$\mathbb{E}(S) = x_0 + \frac{\alpha}{\beta},$$

and the 2nd and 3rd central moments

$$\mathbb{E}(S - \mathbb{E}(S))^2 = \mathbb{V}ar(S) = \frac{\alpha}{\beta^2}, \;\; \mathbb{E}(S - \mathbb{E}(S))^3 = \frac{2\alpha}{\beta^3}.$$

From here,

$$\alpha = 4\frac{[\mathbb{V}ar(S)]^3}{[\mathbb{E}(S - \mathbb{E}(S))^3]^2},$$

$$\beta = 2\frac{\mathbb{V}ar(S)}{\mathbb{E}(S - \mathbb{E}(S))^3},$$

$$x_0 = \mathbb{E}(S) - 2\frac{[\mathbb{V}ar(S)]^2}{\mathbb{E}(S - \mathbb{E}(S))^3}.$$

For compound Poisson distribution,

$$\mathbb{E}(S) = p_1\, \lambda, \;\; \mathbb{V}ar(S) = p_2\, \lambda, \;\; \text{and} \;\; \mathbb{E}(S - \mathbb{E}(S))^3 = \lambda\, p_3.$$

Therefore,

$$\alpha = 4\lambda\, \frac{p_2^3}{p_3^2}, \quad \beta = 2\, \frac{p_2}{p_3}, \;\; \text{and} \;\; x_0 = \lambda\, p_1 - 2\lambda\, \frac{p_2^2}{p_3}.$$

For compound negative binomial,

$$\mathbb{E}(S) = p_1\,\frac{r(1-p)}{p}, \quad \mathbb{V}ar(S) = p_1^2\,\frac{r(1-p)^2}{p^2} + p_2\,\frac{r(1-p)}{p},$$

and

$$\mathbb{E}(S - \mathbb{E}(S))^3 = rp_3\frac{1-p}{p} + 3rp_1p_2\frac{(1-p)^2}{p^2} + 2rp_1^3\frac{(1-p)^3}{p^3}.$$

Example 12.5.1. Assume $N \sim Poi(\lambda = 16)$ and $X_i = 1$. Approximate $\mathbb{P}(S < 25)$ (a) using the normal approximation, (b) using the translated gamma approximation.
SOLUTION: (a)

$$\mathbb{P}(S < 25) = \mathbb{P}\left(Z < \frac{25 - 16 + .5}{\sqrt{16}}\right) = \mathbb{P}(Z < 2.38) = .9912.$$

(b) $\lambda = 16,\ \alpha = 64,\ \beta = 2,\ x_0 = -16,$

$$\mathbb{P}(S < 25) = G(25 + 16 + .5,\, 64,\, 2) = G(41.5,\, 64,\, 2) = .9866.$$

Example. Assume $N \sim NB(r = 10, p = .01)$ and $X_i \sim Unif(0, 1)$. Approximate $\mathbb{P}(S > 600)$.

13.1. Collective Risk Models over an Extended Period. Introduction.

Definition. A stochastic process $\{X(t),\ t \geq 0\}$ is a collection of random variables indexed by t called time.
Definition. Denote by $S(t)$ the aggregate claim process paid in the interval $[0,\, t]$. Let $c(t)$ be the premiums collected in the interval $[0,\, t]$, and u be the company's surplus at time 0. Then the surplus process at time t is $U(t) = u + c(t) - S(t),\ t \geq 0$.
We will assume $c(t) = ct$ where $c > 0$ is a constant premium rate. Let T_i denote the time when the ith claim occurred. Picture of a typical path of $U(t)$, linear growth, jumps at T_i.
Definitions. A ruin occurs when the surplus $U(t)$ becomes negative. Denote the time of ruin $T = \min\{t : t \geq 0 \text{ and } U(t) < 0\}$. Assume $T = \infty$ if $U(t) \geq 0$ for all t. Let $\psi(u) = \mathbb{P}(T < \infty) = \mathbb{P}(U(t) < 0 \text{ for some } t)$ denote the probability of ruin considered as a function of initial surplus.

13.3. A Continuous Time Model. A Compound Poisson Process.

Definition. Consider a portfolio of insurance. Let $\{N(t),\, t \geq 0\}$ denote the claim number process and $\{S(t),\, t \geq 0\}$ denote the aggregate claim process. Let X_i be the amount of the ith claim. Then $S(t) = X_1 + X_2 + \cdots + X_{N(t)}$. Picture $N(t)$ and $S(t)$.

Definition. A stochastic process $\{N(t),\ t \geq 0\}$ is called a counting process

if it represents the total number of events that have occurred up to time t. A counting process is such that (1) $N(t) \geq 0$, (2) $N(t)$ is integer valued, (3) If $s < t$, then $N(s) \leq N(t)$, and (4) For $s < t$, $N(t) - N(s)$ is the number of events that have occurred in the interval (s, t).
Definition. A counting process is said to have independent increments if the number of events occurring in disjoint time intervals are independent.
Definition. A counting process $\{N(t),\, t \geq 0\}$ is called a Poisson process if (1) $N(0) = 0$, (2) it has independent increments, and (3) the number of events in an interval of length t is a Poisson random variable with rate $\lambda\, t$, that is,

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{(\lambda\, t)^n}{n!}\, e^{-\lambda\, t}.$$

Proposition. Let $V_k = T_k - T_{k-1}$, $k > 1$ be the time between two consecutive claims (called the waiting or interarrival or interevent time). If $\{N(t),\, t \geq 0\}$ is a Poisson process, then $V_k \overset{i.i.d.}{\sim} Exp(\beta = 1/\lambda)$ and $T_k \sim Gamma(\alpha = k,\, \beta = 1/\lambda)$.

Definition. If claim amounts X_i are i.i.d. random variables independent of the claim number process $\{N(t),\, t \geq 0\}$ which is assumed to be a Poisson process, then $\{S(t),\, t \geq 0\}$ is a compound Poisson process. The mean and variance of $S(t)$ is $\mathbb{E}(S(t)) = \lambda\, t\, p_1$ and $\mathbb{V}ar(S(t)) = \lambda\, t\, p_2$.

13.4. Ruin Probability and the Claim Amount Distribution.

We make the following assumptions:
(1) $S(t)$ is a compound Poisson process.
(2) $c > \lambda\, p_1$, that is, the premium collection rate exceeds the expected claim payments per unit time.
(3) Define the relative security loading θ by $c = (1+\theta)\, \lambda\, p_1$ where $\theta > 0$. If $\theta \leq 0$, then the ruin is certain and $\psi(u) = 1$ for any initial surplus u.

Definition. The adjustment coefficient R is the positive solution of the equation

$$\lambda\, (M_X(r) - 1) = c\, r,$$

or, equivalently, after substitution $c = (1+\theta)\, \lambda\, p_1$, the positive solution of $1 + (1+\theta)\, p_1\, r = M_X(r)$. Picture.

Example 13.4.1. Find the adjustment coefficient if X is exponential with mean $1/\beta$.
SOLUTION:

$$M_X(r) = \frac{\beta}{\beta - r}, \quad p_1 = 1/\beta, \quad \Longrightarrow \quad 1 + (1+\theta)\frac{r}{\beta} = \frac{\beta}{\beta - r},$$

r solves the quadratic equation $(1+\theta) r^2 - \theta \beta\, r = 0$. One solution is $r = 0$ and the other is

$$R = \frac{\theta\beta}{1+\theta}.$$

Theorem 13.4.1. Let $\{U(t) = u + ct - S(t),\, t \geq 0\}$ be the surplus process, and let $S(t)$ be a compound Poisson process. Assume also that $c > \lambda p_1$. Then for all $u \geq 0$,

$$\psi(u) = \frac{\exp(-Ru)}{\mathbb{E}\left[\exp(-RU(T)) | \, T < \infty\right]}$$

where R is the adjustment coefficient, and T is the time of ruin.

Example 13.4.4. Compute $\psi(u)$ if $X \sim Exp(1/\beta)$.
SOLUTION: Let $\hat{u}$ denote the surplus just before T and let X be the claim amount that caused the ruin. Then for any $y > 0$,

$$\mathbb{P}(U(T) < -y | \, T < \infty) = \mathbb{P}(X > \hat{u} + y | \, X > \hat{u}) = \frac{\mathbb{P}(X > \hat{u} + y)}{\mathbb{P}(X > \hat{u})}$$

$$= \frac{\beta \int_{\hat{u}+y}^{\infty} e^{-\beta x}\, dx}{\beta \int_{\hat{u}}^{\infty} e^{-\beta x}\, dx} = e^{-\beta y}.$$

Hence, the conditional distribution of $-U(T)$, given $T < \infty$, is $Exp(1/\beta)$, and so,

$$\mathbb{E}\left[\exp(-RU(T)) | \, T < \infty\right] = \frac{\beta}{\beta - R}.$$

Finally,

$$\psi(u) = \frac{(\beta - R)\, e^{-Ru}}{\beta} = \frac{1}{1+\theta} \exp\left(-\frac{\theta \beta u}{1+\theta}\right).$$

13.5. The First Surplus Below the Initial Level.

Theorem. Consider the surplus process $\{U(t),\, t \geq 0\}$ based on a compound Poisson aggregate claim process $\{S(t),\, t \geq 0\}$. Let $X_i \sim P(x)$. Denote by L_1 the amount by which the surplus falls below the initial level u for the first time, given that this ever happens. Then $f_{L_1}(y) = \frac{1}{p_1}[1 - P(y)],\ y > 0$.
Remark. Notice that the density integrates to one, since $p_1 = \mathbb{E}(X) = \int_0^\infty xp(x)\, dx = \int_0^\infty [1 - P(x)]\, dx$.

Example 13.5.2. Find $f_{L_1}(y)$ if $X_i \sim Exp(1/\beta)$.
SOLUTION: $f_{L_1}(y) = \beta\,[1 - (1 - e^{-\beta y})] = \beta\, e^{-\beta y}$, that is, $L \sim Exp(1/\beta)$.

13.6. The Maximal Aggregate Loss.

Definition. A maximal aggregate loss is $L = \max_{t \geq 0}\{S(t) - ct\}$, the maximal excess of aggregate claim over received premiums. Since $L = 0$ for $t = 0$, it follows that $L \geq 0$.

Proposition. The c.d.f. of L is $\mathbb{P}(L \leq x) = 1 - \psi(x),\ \ x \geq 0$.
PROOF: For $u \geq 0$, write $1 - \psi(u) = \mathbb{P}(U(t) \geq 0 \text{ for all } t) = \mathbb{P}(u + ct - S(t) \geq 0 \text{ for all } t) = \mathbb{P}(S(t) - ct \leq u \text{ for all } t) = \mathbb{P}(L \leq u)$.

Also, $\mathbb{P}(L < u) = 0$ if $u < 0$, and $\mathbb{P}(L \le 0) = \{L \ge 0\} = \mathbb{P}(L = 0) = 1 - \psi(0) = \theta/(1+\theta)$. Thus, L has a mixed distribution with point mass $\theta/(1+\theta)$ at zero. Picture.

From the proposition it follows that if we know the distribution of L, we can find the probability of ruin.
Theorem 13.6.1.

$$M_L(r) = \frac{\theta\, p_1\, r}{1 + (1+\theta)\, p_1\, r - M_X(r)}.$$

Proposition.

$$M_L(r) = 1 - \psi(0) + \int_0^\infty e^{ur}[-\psi'(u)]\, du = \frac{\theta}{1+\theta} + \int_0^\infty e^{ur}[-\psi'(u)]\, du,$$

hence,

$$\int_0^\infty e^{ur}[-\psi'(u)]\, du = \frac{1}{1+\theta}\, \frac{\theta[M_X(r) - 1]}{1 + (1+\theta)\, p_1\, r - M_X(r)}.$$

Example 13.6.1. Suppose $X_i \sim Exp(1/\beta)$. Find $\psi(u)$.
SOLUTION: $M_X(r) = \frac{\beta}{\beta - r}$, therefore,

$$\int_0^\infty e^{ur}[-\psi'(u)]\, d = C\, \frac{r_1}{r_1 - r}$$

where $C = 1/(1+\theta)$ and $r_1 = (\theta\,\beta)/(1+\theta)$. Check that $\psi(u) = C\, e^{-r_1\, u}$ as expected.

Example 13.6.2. Suppose $\theta = 2/5$ and $X \sim p(x) = (3/2)e^{-3x} + (7/2)e^{-7x},\ x > 0$. Find $\psi(u)$.
SOLUTION: $M_X(r) = \frac{3/2}{3-r} + \frac{7/2}{7-r}$ and $p_1 = 5/21$. Thus,

$$\int_0^\infty e^{ur}[-\psi'(u)]\, du = \frac{24/35}{1-r} + \frac{6/35}{6-r},$$

and therefore $\psi(u) = (24/35)e^{-u} + (1/35)\, e^{-6u},\ u \ge 0$.

Chapter 3. Survival Distributions and Life Tables.
The models we studied before can be used to model any kind of insurance. Now we will study notions related to life insurance.

3.2. Probability for the Age-at-Death.

Definition. Let X be the age-at-death of a person. X is a continuous random variable with c.d.f. $F_X(x)$. The survival function is $s(x) = 1 - F_X(x) = \mathbb{P}(X > x)$. For any $x > 0$, $s(x)$ represents the probability that a newborn will attain age x.

Traditionally, the survival function $s(x)$ plays a fundamental role in actuarial science. In probability theory $F_X(x)$ plays this role. For example, the probability that a newborn dies between ages x and $y > x$ is

$$\mathbb{P}(x < X \le y) = F_X(y) - F_X(x) = s(x) - s(y).$$

As another example, the conditional probability that a newborn will die between ages x and $y > x$, given that he survived till age x is

$$\mathbb{P}(x < X \le y | \, X > x) = \frac{F_X(y) - F_X(x)}{1 - F_X(x)} = \frac{s(x) - s(y)}{s(x)}.$$

NOTATION. The symbol (x) is used to denote a life-age-x. The future lifetime of (x) is denoted by $T(x) = X - x$. The distribution function of $T(x)$ is ${}_tq_x = \mathbb{P}(T(x) \le t),\ t \ge 0$, which can be interpreted as the probability that (x) will die within t years. The probability that (x) will attain age $x + t$ is ${}_tp_x = 1 - {}_tq_x = \mathbb{P}(T(x) > t),\ t \ge 0$. The function ${}_tp_x$ is the survival function for (x). Notice that ${}_tp_0 = s(t)$.

Further, denote by $q_x = {}_1q_x = \mathbb{P}(T(x) \le 1)$ the probability that (x) will die within one year, and $p_x = {}_1p_x$ the probability that (x) will attain age $x + 1$. Also, ${}_{t|\,u}q_x = \mathbb{P}(t < T(x) < t + u) = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x$. If $u = 1$, ${}_{t|\,u}q_x = {}_{t|}q_x$.

Useful formulas.

$${}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0} = \frac{s(x+t)}{s(x)}, \quad {}_tq_x = 1 - \frac{s(x+t)}{s(x)}, \quad {}_{t|\,u}q_x = \frac{s(x+t) - s(x+t+u)}{s(x)}$$

$$= \left[\frac{s(x+t)}{s(x)}\right] \left[\frac{s(x+t) - s(x+t+u)}{s(x+t)}\right] = {}_tp_x \, {}_uq_{x+t}.$$

Definition. The number of future years completed by (x) prior to death is a discrete random variable called the curtate-future-lifetime of (x) and is denoted by $K(x)$. We have

$$\mathbb{P}(K(x) = k) = \mathbb{P}(k \le T(x) < k + 1) = {}_kp_x - {}_{k+1}p_x = {}_kp_x q_{x+k} = {}_{k|}q_x.$$

The distribution function of $K(x)$ is $F_{K(x)}(n) = \sum_{k=0}^{n} \mathbb{P}(k \le T(x) < k+1) = \mathbb{P}(T(x) < n + 1) = {}_{n+1}q_x$.

Note. Sometimes instead of $T(x)$ and $K(x)$ we will write T and K.

Definition. The probability $\mathbb{P}(x < X \le x + dx) | \, X > x)$ is the conditional probability that a person will die within dx given that he attained age x. This probability can be written as

$$\mathbb{P}(x < X \le x + dx | \, X > x) = \frac{F_X(x + dx) - F_x(x)}{1 - F_X(x)} \approx \frac{f_X(x)}{1 - F_X(x)} \, dx.$$

The function $\frac{f_X(x)}{1-F_X(x)}$ is a conditional probability density of X at age x given survival to this age. This function is denoted by

$$\mu(x) = \frac{f_X(x)}{1 - F_X(x)} = -\frac{s'(x)}{s(x)}$$

and is called in actuarial science the force of mortality. In survival analysis and reliability theory this function is denoted by $h(x)$ and is called the failure rate or hazard rate function.

Useful formulas. From the definition of $\mu(x)$,

(1) $s(x) = e^{-\int_0^x \mu(y)\,dy}$,
(2) $F_X(x) = 1 - s(x) = 1 - e^{-\int_0^x \mu(y)\,dy}$,
(3) $f_X(x) = F'_X(x) = e^{-\int_0^x \mu(y)\,dy}\,\mu(x) = s(x)\,\mu(x) = {}_xp_0\,\mu(x)$,
(4) $f_{T(x)}(t) = F'_{T(x)}(t) = \frac{d_t q_x}{dt} = \frac{d}{dt}\left[1 - \frac{s(x+t)}{s(x)}\right] = -\frac{s'(x+t)}{s(x)}$
$= \left[\frac{s(x+t)}{s(x)}\right]\left[-\frac{s'(x+t)}{s(x+t)}\right] = {}_tp_x\,\mu(x+t)$.

3.3 Life Tables.

Definition. A life table is a published summary of the distribution of the age-at-death random variable X and related functions.
NOTATION. Consider a group of l_0 newborns, each with survival function $s(x)$. This group is called a cohort or a random survivorship group. Let $\mathfrak{L}(x)$ be the group's number of survivors at age x. Let $l_x = \mathbb{E}[\mathfrak{L}(x)] = l_0\mathbb{P}($a person survives to age $x) = l_0\,s(x)$. The function l_x represents the expected number of survivors to age x from the group of l_0 newborns.
Also, denote by ${}_n\mathfrak{D}_x$ the number of deaths between ages x and $x+n$ from the l_0 newborns. Let ${}_nd_x = \mathbb{E}[{}_n\mathfrak{D}_x] = l_0[s(x) - s(x+n)] = l_x - l_{x+n}$. When $n = 1$, we write ${}_1d_x = d_x$ and ${}_1\mathfrak{D}_x = \mathfrak{D}_x$.
Useful Formulas:
(1) Since $l_x = l_0 s(x)$, the following relation holds:

$$-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{1}{s(x)}\frac{ds(x)}{dx} = \mu(x).$$

(2) From above expression, $l_x = l_0 \exp\left[-\int_0^x \mu(y)\,dy\right]$.
(3) From above expression, $l_{x+n} = l_x \exp\left[-\int_x^{x+n} \mu(y)\,dy\right]$.
(4) From above expression, ${}_nd_x = l_x - l_{x+n} = \int_x^{x+n} l_y\,\mu(y)\,dy$.

Definition. The limiting age ω is the age for which $s(x) > 0$ for all $x < \omega$ and $s(x) = 0$ for all $x \geq \omega$.

Section 3.3.2. Life Table Example. Study the table on pages 60 – 63, go over observations on page 63 and Example 3.3.1 on page 64.

4.2.1 – 4.2.3. Insurances Payable at the Moment of Death.

Definition. The benefit function is the claim payment and is denoted by b_t.

Definition. The discount function, denoted by v_t, is the interest discount factor for the time interval of length t from the time of payment back to the time of policy issue.

Definition. For life insurances $t = T = T(x)$, the insured's future-lifetime. Thus, the present value function is $Z = z_T = b_T\, v_T$.

Definition. The discount function has the form $v_t = e^{-\delta_t}$ where δ_t is called the force of interest.

Definition. The n-year term life insurance is the insurance that provides a payment only if an insured dies within n years of policy issue.
In this case

$$b_T = \begin{cases} 1 & \text{if } T \le n \\ 0 & \text{if } T > n, \end{cases} \quad v_T = v^T, \quad Z = \begin{cases} v^T & \text{if } T \le n \\ 0 & \text{if } T > n. \end{cases}$$

Definition. The expectation of the present value of the payments, $\mathbb{E}(Z)$, is called the actuarial present value. For the n-year term life insurance it is denoted by

$$\bar{A}^{1}_{x:\overline{n|}} = \mathbb{E}(Z) = \mathbb{E}(z_T) = \int_0^\infty z_t\, f_T(t)\, dt$$
$$= \int_0^n v^t\, {}_tp_x\, \mu(x+t)\, dt = \int_0^n e^{-\delta\, t}\, {}_tp_x\, \mu(x+t)\, dt.$$

Example. $T \sim Unif(0,\theta)$. Then $\bar{A}^{1}_{x:\overline{n|}} = \int_0^n e^{-\delta\, t} \frac{1}{\theta}\, dt = \frac{1}{\delta\, \theta}\Big(1 - e^{-\delta\, n}\Big)$. Suppose $\theta = 30, \delta = 0.02$, and $n = 10$. For a 45-year-old person, $\bar{A}^{1}_{45:\overline{10|}} = \frac{1}{(0.02)(30)}\Big(1 - e^{-(0.02)(10)}\Big) = 0.3021$.

Example. $T \sim Exp(mean = 1/\mu)$. Then $\bar{A}^{1}_{x:\overline{n|}} = \int_0^n e^{-\delta\, t}\, \mu\, e^{-\mu\, t}\, dt = \frac{\mu}{\mu+\delta}\Big(1 - e^{-(\mu+\delta)\, n}\Big)$. Suppose $\delta = 0.02, \mu = 0.06$, and $n = 10$. For a 45-year-old person, $\bar{A}^{1}_{45:\overline{10|}} = \frac{0.06}{0.06+0.02}\Big(1 - e^{-(0.06+0.02)(10)}\Big) = 0.4130$.

Definition. The whole life insurance is the insurance that provides a payment following the insured's death that happens at any time in the future.

In this case

$$b_T = 1, \;\; v_T = v^T, \;\text{ and }\; Z = v^T, \;\; T \ge 0.$$

The actuarial present value is

$$\bar{A}_x = \mathbb{E}(Z) = \int_0^\infty v^t\, {}_tp_x\, \mu(x+t)\, dt = \int_0^\infty e^{-\delta\, t}\, {}_tp_x\, \mu(x+t)\, dt.$$

Example. $T \sim Unif(0,\theta)$. Then $\bar{A}_x = \int_0^\theta e^{-\delta\, t}\frac{1}{\theta}\, dt = \frac{1}{\delta\, \theta}\Big(1 - e^{\delta\theta}\Big)$. Suppose $\theta = 30$, and $\delta = 0.02$. For a 45-year-old person, $\bar{A}_{45} = \frac{1}{(0.02)(30)}\Big(1 -$

$e^{-(0.02)(30)}\Big) = 0.6820.$

EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then $\bar{A}_x = \int_0^\infty e^{-\delta t}\,\mu\, e^{-\mu t}\, dt = \frac{\mu}{\mu+\delta}$. Suppose $\delta = 0.02$, and $\mu = 0.06$. For a 45-year-old person, $\bar{A}_{45} = \frac{0.06}{0.06+0.02} = 0.75$.

Definition. The <u>n-year pure endowment</u> is an insurance that provides a payment at the end of the n years if and only the insured is still alive.
In this case,

$$b_T = \begin{cases} 0 & \text{if } T \le n \\ 1 & \text{if } T > n, \end{cases} \quad v_t = v^n, \text{ and } Z = \begin{cases} 0 & \text{if } T \le n \\ v^n & \text{if } T > n. \end{cases}$$

The actuarial present value is

$$A_{x:\overline{n|}}^{\;\;1} = \mathbb{E}(Z) = v^n \mathbb{P}(T > n) = v^n\, {}_n p_x.$$

EXAMPLE. $T \sim Unif(0,\theta)$. Then $A_{x:\overline{n|}}^{\;\;1} = e^{-\delta n}\left(1 - \frac{n}{\theta}\right)$. Suppose $\theta = 30, \delta = 0.02$, and $n = 10$. For a 45-year-old person, $A_{45:\overline{10|}}^{\;\;1} = e^{-(0.02)(10)}\Big(1 - \frac{10}{30}\Big) = 0.5458$.
EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then $A_{x:\overline{n|}}^{\;\;1} = e^{-\delta n}\, e^{-\mu n} = e^{-(\mu+\delta)n}$. Suppose $\delta = 0.02, \mu = 0.06$, and $n = 10$. For a 45-year-old person, $A_{45:\overline{10|}}^{\;\;1} = e^{-(0.06+0.02)(10)} = 0.4493$.

Definition. The <u>n-year endowment insurance</u> is an insurance that provides a payment either following the death of the insured or at the end of the n years if the insured is still alive, whichever occurs first.
In this case,

$$b_t = 1, \;\; v_T = \begin{cases} v^T & \text{if } T \le n \\ v^n & \text{if } T > n \end{cases}, \text{ and } Z = \begin{cases} v^T & \text{if } T \le n \\ v^n & \text{if } T > n. \end{cases}$$

The actuarial present value is

$$\bar{A}_{x:\overline{n|}} = \bar{A}_{x:\overline{n|}}^{1} + A_{x:\overline{n|}}^{\;\;1} = \int_0^n e^{-\delta t}\, {}_t p_x\, \mu(x+t)\, dt + v^n\, {}_n p_x.$$

EXAMPLE. $T \sim Unif(0,\theta)$. Then $\bar{A}_{x:\overline{n|}} = \frac{1}{\delta\,\theta}\left(1 - e^{-\delta n}\right) + e^{-\delta n}\left(1 - \frac{n}{\theta}\right)$. Suppose $\theta = 30, \delta = 0.02$, and $n = 10$. For a 45-year-old person, $\bar{A}_{45:\overline{10|}} = \frac{1}{(0.02)\,(30)}\left(1 - e^{-(0.02)(10)}\right) + e^{-(0.02)(10)}\left(1 - \frac{10}{30}\right) = 0.3021 + 0.5458 = 0.8479$.
EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then $\bar{A}_{x:\overline{n|}} = \frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)\,n}\right) + e^{-(\mu+\delta)n}$. Suppose $\delta = 0.02, \mu = 0.06$, and $n = 10$. For a 45-year-old person,

$\bar{A}_{45:\overline{10}|} = \dfrac{0.06}{0.06+0.02}\left(1-e^{-(0.06+0.02)(10)}\right)+e^{-(0.06+.02)(10)} = 0.4130+0.4493 = 0.8623.$

Definition. The <u>m-year deferred insurance</u> is an insurance that provides a payment following the death of the insured only if he dies at least m years after the policy is issued.
In this case,

$$b_T = \begin{cases} 1 & \text{if } T > m \\ 0 & \text{if } T \le m \end{cases}, \quad v_T = v^T, \quad \text{and} \quad Z = \begin{cases} v^T & \text{if } T > m \\ 0 & \text{if } T \le m. \end{cases}$$

The actuarial present value is

$$_{m|}\bar{A}_x = \int_m^\infty v^t \, _tp_x \, \mu(x+t)\, dt.$$

Example. $T \sim Unif(0,\theta)$. Then $_{m|}\bar{A}_x = \displaystyle\int_m^\theta e^{-\delta t}\frac{1}{\theta}\, dt = \frac{1}{\delta\,\theta}\left(e^{-\delta\, m} - e^{-\delta\,\theta}\right)$. Suppose $\theta = 30, \delta = 0.02$, and $m = 5$. For a 45-year-old person, $_{5|}\bar{A}_{45} = \dfrac{1}{(0.02)(30)}\left(e^{-(0.02)\,(5)} - e^{-(0.02)\,(30)}\right) = 0.5934.$

Example. $T \sim Exp(mean = 1/\mu)$. Then $_{m|}\bar{A}_x = \displaystyle\int_m^\infty e^{-\delta\, t}\, \mu\, e^{-\mu\, t}\, dt = \frac{\mu}{\mu+\delta}\, e^{-(\mu+\delta)\, m}$. Suppose $\delta = 0.02, \mu = 0.06$, and $m = 5$. For a 45-year-old person, $_{5|}\bar{A}_{45} = \dfrac{0.06}{0.06+0.02}\, e^{-(0.06+0.02)(5)} = 0.5027.$

Definition. The <u>m-year deferred n-year term insurance</u> is an insurance that provides a payment following the death of the insured only if he dies in the period between m and $m+n$ year after the policy is issued.
In this case,

$$b_T = \begin{cases} 1 & \text{if } m < T \le m+n \\ 0 & \text{if } T \le m \text{ or } T > m+n \end{cases}, \quad v_T = v^T,$$

and

$$Z = \begin{cases} v^T & \text{if } m < T \le m+n \\ 0 & \text{if } T \le m \text{ or } T > m+n. \end{cases}$$

The actuarial present value is

$$_{m|n}\bar{A}_x = \int_m^{m+n} v^t \, _tp_x \, \mu(x+t)\, dt.$$

Example. $T \sim Unif(0,\theta)$. Then $_{m|n}\bar{A}_x = \displaystyle\int_m^{m+n} e^{-\delta\, t}\frac{1}{\theta}\, dt = \frac{1}{\delta\,\theta}\left(e^{-\delta\, m} - e^{-\delta\,(m+n)}\right) = \frac{1}{\delta\,\theta}\, e^{-\delta\, m}\left(1 - e^{-\delta\, n}\right)$. Suppose $\theta = 30, \delta = 0.02, m = 5$, and $n =$

10. For a 45-year-old person, ${}_{5|10}\bar{A}_{45} = \dfrac{1}{(0.02)(30)}\, e^{-(0.02)(5)}\Big(1 - e^{-(0.02)(10)}\Big) =$ 0.2734.

EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then ${}_{m|n}\bar{A}_x = \displaystyle\int_m^{m+n} e^{-\delta\, t}\, \mu\, e^{-\mu\, t}\, dt =$ $\dfrac{\mu}{\mu+\delta}\, e^{-(\mu+\delta)\, m}\Big(1 - e^{-(\mu+\delta)\, n}\Big)$. Suppose $\delta = 0.02, \mu = 0.06, m = 5,$ and $n = 10$. For a 45-year-old person, ${}_{5|10}\bar{A}_{45} = \dfrac{0.06}{0.06+0.02}\, e^{-(0.06+0.02)(5)}\Big(1 - e^{-(0.06+0.02)(10)}\Big) = 0.2768$.

5.1. Life Annuities. Introduction.

Definition. A life annuity is a series of payments made continuously or at equal intervals (such as monthly, yearly) while the insured survives.

Definition. Denote by $\bar{a}_{\overline{t}|}$ the present value of payments to be made at a future time t. Then from compound interest theory, the following relation holds $1 = \delta \bar{a}_{\overline{t}|} + v^t$. It can be interpreted as indicating that a unit amount invested now will produce annual interest δ payable continuously for t years at which point interest ceases and the investment is repaid.

5.2. Continuous Life Annuities.

Definition. A whole life annuity is a series of payments payable continuously while (x) survives. The present value is

$$Y = \bar{a}_{\overline{T}|},$$

and the actuarial present value is

$$\bar{a}_x = \mathbb{E}(Y) = \int_0^\infty \bar{a}_{\overline{t}|}\, {}_t p_x\, \mu(x+t)\, dt = \int_0^\infty v^t\, {}_t p_x\, dt.$$

EXAMPLE. $T \sim Unif(0, \theta)$. Then

$$\bar{a}_x = \int_0^\theta e^{-\delta\, t}\Big(1 - \frac{t}{\theta}\Big)\, dt = \frac{1}{\delta} - \frac{1}{\delta^2\, \theta}\Big(1 - e^{-\delta\, \theta}\Big).$$

Suppose $\theta = 30$, and $\delta = 0.02$. For a 45-year-old person,

$$\bar{a}_{45} = \frac{1}{0.02} - \frac{1}{(0.02)^2(30)}\Big(1 - e^{-(0.02)(30)}\Big) = 12.40.$$

EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then

$$\bar{a}_x = \int_0^\infty e^{-\delta\, t}\, e^{-\mu\, t}\, dt = \frac{1}{\mu+\delta}.$$

Suppose $\delta = 0.02$, and $\mu = 0.06$. For a 45-year-old person, $\bar{a}_{45} = \dfrac{1}{0.06 + 0.02} = 12.50$.

Definition. The <u>n-year temporary life annuity</u> is a series of payments payable continuously while (x) survives during the next n years. The present value is

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & \text{if } 0 \le T < n \\ \bar{a}_{\overline{n}|} & \text{if } T \ge n \end{cases},$$

and the actuarial present value is

$$\bar{a}_{x:\overline{n}|} = \mathbb{E}(Y) = \int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt + \bar{a}_{\overline{n}|}\, {}_np_x = \int_0^n v^t\, {}_tp_x\, dt.$$

Example. $T \sim Unif(0, \theta)$. Then

$$\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\Big(1 - \frac{t}{\theta}\Big)\, dt = \frac{1}{\delta} - \frac{e^{-\delta n}}{\delta}\Big(1 - \frac{n}{\theta}\Big) - \frac{1}{\delta^2 \theta}\Big(1 - e^{-\delta n}\Big).$$

Suppose $\theta = 30, \delta = 0.02$, and $n = 10$. For a 45-year-old person,

$$\bar{a}_{45:\overline{10}|} = \frac{1}{0.02} - \frac{e^{-(0.02)(10)}}{0.02}\Big(1 - \frac{10}{30}\Big) - \frac{1}{(0.02)^2\,(30)}\Big(1 - e^{-(0.02)(10)}\Big) = 7.60.$$

Example. $T \sim Exp(mean = 1/\mu)$. Then

$$\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\, e^{-\mu t}\, dt = \frac{1}{\mu + \delta}\Big(1 - e^{-(\mu+\delta)\, n}\Big).$$

Suppose $\delta = 0.02, \mu = 0.06$ and $n = 10$. For a 45-year-old person, $\bar{a}_{45:\overline{10}|} = \dfrac{1}{0.06 + 0.02}\Big(1 - e^{-(0.06+0.02)(10)}\Big) = 6.88$.

Definition. The <u>n-year deferred whole life annuity</u> is a series of payments payable continuously while (x) survives provided he survives at least n years. The present value is

$$Y = \begin{cases} 0 & \text{if } 0 \le T < n \\ v^n\, \bar{a}_{\overline{T-n}|} = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & \text{if } T \ge n \end{cases},$$

and the actuarial present value is

$${}_{n|}\bar{a}_x = \mathbb{E}(Y) = \int_n^\infty v^n \bar{a}_{\overline{t-n}|}\, {}_tp_x\, \mu(x+t)\, dt = \int_n^\infty v^t\, {}_tp_x\, dt.$$

Example. $T \sim Unif(0, \theta)$. Then

$${}_{n|}\bar{a}_x = \int_n^\theta e^{-\delta t}\Big(1 - \frac{t}{\theta}\Big)\, dt = \frac{e^{-\delta n}}{\delta}\Big(1 - \frac{n}{\theta}\Big) - \frac{1}{\delta^2 \theta}\Big(e^{-\delta n} - e^{-\delta \theta}\Big).$$

Suppose $\theta = 30, \delta = 0.02$, and $n = 5$. For a 45-year-old person,

$$_{5|}\bar{a}_{45} = \frac{e^{-(0.02)(5)}}{0.02}\Big(1 - \frac{5}{30}\Big) - \frac{1}{(0.02)^2\,(30)}\Big(e^{-(0.02)(5)} - e^{-(0.02)(30)}\Big) = 8.03.$$

EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then

$$_{n|}\bar{a}_x = \int_n^\infty e^{-\delta\,t}\,e^{-\mu\,t}\,dt = \frac{1}{\mu+\delta}\,e^{-(\mu+\delta)\,n}.$$

Suppose $\delta = 0.02, \mu = 0.06$ and $n = 5$. For a 45-year-old person, $_{5|}\bar{a}_{45} = \dfrac{1}{0.06+0.02}\,e^{-(0.06+0.02)(5)} = 8.38.$

Definition. The <u>n-year certain and life annuity</u> is a whole life annuity with a guarantee of payments for the first n years. The present value is

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & \text{if } T \le n \\ \bar{a}_{\overline{T}|} & \text{if } T > n \end{cases},$$

and the actuarial present value is

$$\bar{a}_{\overline{x:\,\overline{n}|}} = \mathbb{E}(Y) = \int_0^n \bar{a}_{\overline{n}|}\,{}_tp_x\,\mu(x+t)\,dt + \int_n^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\,\mu(x+t)\,dt = \bar{a}_{\overline{n}|} + \int_n^\infty v^t\,{}_tp_x\,dt.$$

EXAMPLE. $T \sim Unif(0,\theta)$. Then

$$\bar{a}_{\overline{x:\,\overline{n}|}} = \frac{1-e^{-\delta\,n}}{\delta} + \int_n^\theta e^{-\delta\,t}\Big(1 - \frac{t}{\theta}\Big)\,dt = \frac{1}{\delta}\Big(1 - \frac{n}{\theta}\,e^{-\delta\,n}\Big) - \frac{1}{\delta^2\,\theta}\Big(e^{-\delta\,n} - e^{-\delta\,\theta}\Big).$$

Suppose $\theta = 30, \delta = 0.02$, and $n = 5$. For a 45-year-old person,

$$\bar{a}_{\overline{45:\,\overline{5}|}} = \frac{1}{0.02}\Big(1 - \frac{5}{30}\,e^{-(0.02)(5)}\Big) - \frac{1}{(0.02)^2\,(30)}\Big(e^{-(0.02)(5)} - e^{-(0.02)(30)}\Big) = 12.79.$$

EXAMPLE. $T \sim Exp(mean = 1/\mu)$. Then

$$\bar{a}_{\overline{x:\,\overline{n}|}} = \frac{1-e^{-\delta\,n}}{\delta} + \int_n^\infty e^{-\delta\,t}\,e^{-\mu\,t}\,dt = \frac{1-e^{-\delta\,n}}{\delta} + \frac{1}{\mu+\delta}\,e^{-(\mu+\delta)\,n}.$$

Suppose $\delta = 0.02, \mu = 0.06$ and $n = 5$. For a 45-year-old person, $\bar{a}_{\overline{45:\,\overline{5}|}} = \frac{1-e^{-(0.02)(5)}}{0.02} + \frac{1}{0.06+0.02}\,e^{-(0.06+0.02)(5)} = 13.14.$

6.1. Benefit Premiums. Introduction.

Definition. The <u>premium principle</u> is a principle under which an insurance premium is determined.

Definition. The <u>insurer's loss</u> is the random variable of the present value of benefits to be paid by the insurer minus the annuity of premiums to be paid by the insured.

Definition. The percentile premium principle (principle I) is the premium principle under which the premium is the least annual premium such that the insurer has probability of a positive financial loss of at most a certain percentage.

Definition The equivalence principle (principle II) is the premium principle under which the premium is such that the present value of the expected loss to the insurer at policy issue equals zero.

Definition. The exponential premium principle (principle III) is the premium principle under which the premium is such that the insurer with an exponential utility function is indifferent between accepting or rejecting the risk.

Definition. The percentile premiums are the premiums established using the percentile premium principle.
Definition. The benefit premiums are the premiums established using the equivalence principle.
Definition. The exponential premiums are the premiums established using the exponential premium principle.

Example 6.1.1. An insurer is planning to issue a policy to a life-age-0 whose remaining lifetime (in years completed by death) K has distribution ${}_{k|}q_0 = .2,\ \ k = 0, 1, 2, 3, 4$. The policy will pay one unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Find the annual premium P, as determined by:
(a) principle I: P will have the smallest value such that the insurer has probability of a positive loss of at most .25.
(b) principle II: P will have the value such that the present value of the insurer's expected loss is zero.
(c) principle III: P will have the value such that the insurer, using the utility function $u(x) = -e^{-.1x}$, will be indifferent between accepting or rejecting the risk. Assume $i = .06$.
Solution: The present value of the loss to the insurer at policy issue is a random variable $L = v^{K+1} - P\,(1 + v + \cdots + v^K) = v^{K+1} - P\,\frac{1-v^{K+1}}{1-v}$.

(a) We must have $\mathbb{P}(L > 0) \leq .25$. As k increases, the function $l(k) = v^{k+1} - P\,\frac{1-v^{k+1}}{1-v}$ decreases. Since $\mathbb{P}(K = 0) = .2 < .25$, but $\mathbb{P}(K = 0$ or $1) = .4 > .25$, we must have $l(0) > 0$, but $l(1) = 0$. Thus, we must find P such that $l(1) = v^2 - P(1 + v) = 0$. Thus, $P = v^2/(1 + v)$ where $v = 1/(1 + i) = .9434$. Therefore, $P = .45796$.

(b) We want to find P such that $\mathbb{E}(L) = 0$ or $(v - P) + (v^2 - P(1+v)) + (v^3 - P(1+v+v^2)) + (v^4 - P(1+\cdots+v^3)) + (v^5 - P(1+\cdots+v^4)) = 0$. From here,

$$P = \frac{v(1-v^5)}{(1-v)(5+4v+3v^2+2v^3+v^4)} = .3027.$$

(c) Recall that we want to find P such that $u(w) = \mathbb{E}(u(w-L))$ where w denotes the wealth. We have $u(w) = -e^{-.1w} = \mathbb{E}[-e^{-.1(w-L)}]$ or $\mathbb{E}\, e^{.1L} = 1$. Equivalently,

$$e^{.1(v-P)} + e^{.1(v^2-P(1+v))} + e^{.1(v^3-P(1+v+v^2))} + e^{.1(v^4-P(1+\cdots+v^3))} + e^{.1(v^5-P(1+\cdots+v^4))} = \frac{1}{.2}$$

Thus, $P = .3063$.

The above example shows that, in general, to apply the premium principles, one has to do the following:
1. Principle I:
(a) determine the value of k corresponding to the specified percentage;
(b) for the fixed k, solve $v^{k+1} - P(1+v+\cdots+v^k) = 0$. The solution is

$$P = \frac{v^{k+1}(1-v)}{1-v^{k+1}}.$$

2. Principle II:
Find P such that $\mathbb{E}(L) = \mathbb{E}(v^{K+1}) - P\,\mathbb{E}\left(\frac{1-v^{K+1}}{1-v}\right) = 0$. The solution is

$$P = \frac{(1-v)\,\mathbb{E}(v^{K+1})}{1-\mathbb{E}(v^{K+1})}.$$

3. Principle III:
Find P such that

$$\mathbb{E}\left(e^{\alpha L}\right) = 1 \text{ or, equivalently, } \mathbb{E}\left(\exp\left\{\alpha\left(v^{K+1} - P\frac{1-v^{K+1}}{1-v}\right)\right\}\right) = 1$$

where α is the parameter of the exponential utility function $u(w) = -e^{-\alpha w}$.