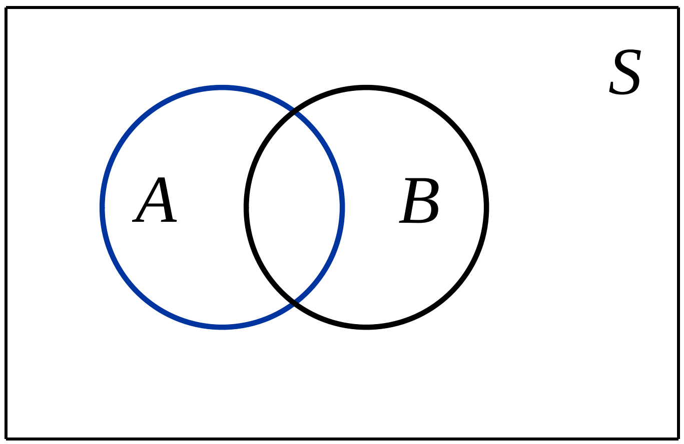
Venn Diagram

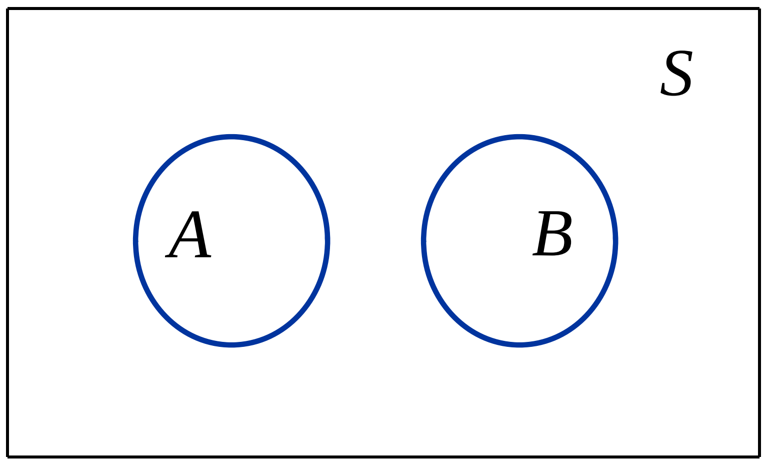
Definition. A **Venn diagram** depicts the sample space S as a rectangle and any event as a circle within this rectangle.

Example. In general, the Venn diagram for any two events A and B looks like this:

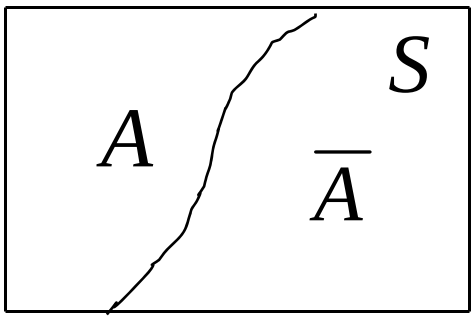


S
A
B

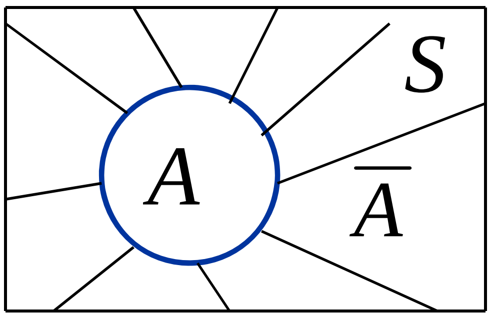
Example. The Venn diagram for two disjoint events A and B looks like this:



Example. The Venn diagram for an event A and its complement $\bar{A}$ can be drawn like this:



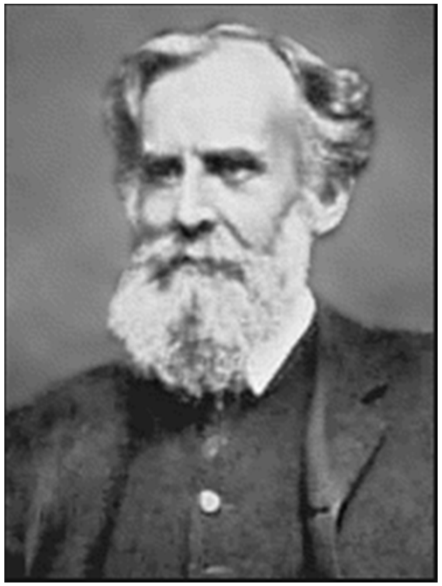
or



Historical Note

John Venn (1834 –1923) was a British logician and philosopher.

He introduced the diagram in 1880 .

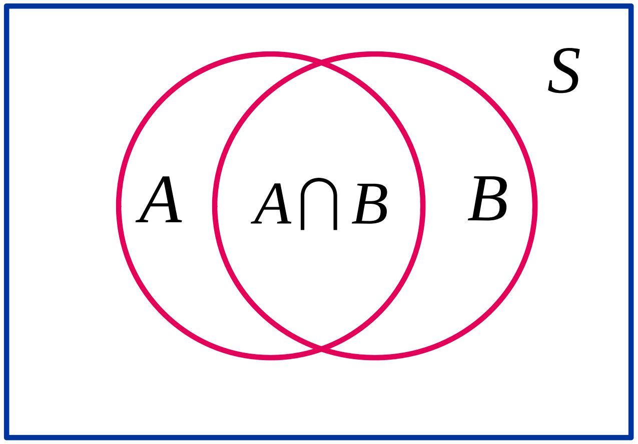


4.4 Intersection of Events and the Multiplication Rule

Definition. The **intersection** of two events consists of the outcomes that are common to both events.

Notation. The intersection of events A and B is denoted by $A \cap B$ (read "A intersect B").

On the Venn diagram, the intersection of A and B looks like this:

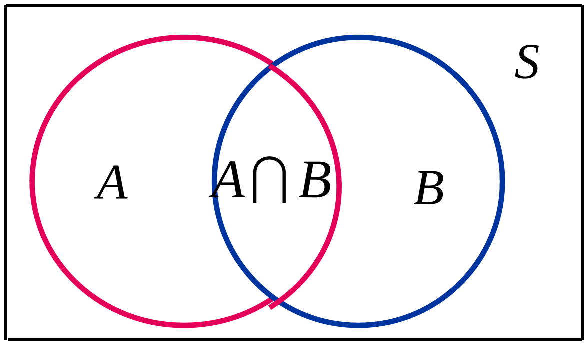


Definition. The probability of the intersection of two events $P(A \cap B)$ is called the **joint probability** of events A and B.

Definition. The **conditional probability** of an event A given event B is computed according to the formula

$$P(A|B)=\frac{P(A\bigcap B)}{P(B)}.$$

To see why it is so, consider the Venn diagram



If B happens, we reduce the sample space to B only. The event A can now happen only if $A \cap B$ happens, and thus the conditional probability $P(A|B)$ is equal to the ratio $\frac{P(A \cap B)}{P(B)}$.

Example. One card is drawn from a standard deck of cards. Given that the card is black, compute the conditional probability that it is an ace.

Solution: Let *A= an ace is drawn*, and *B=a black card is drawn*. The intersection $A \cap B$ = *a black ace is drawn.* We have

$$P(A \cap B) = 2/52, \text{ and } P(B) = 26/52.$$

Hence, $P(A\,|\,B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{2/52}{26/52} = \dfrac{2}{26} = \dfrac{1}{13}.$

Reduction of sample space gives us an alternative solution.

Knowing that a black card is drawn, we can reduce the sample space to the 26 black cards. Now the probability that an ace is drawn is computed as

$$P(A|B) = 2/26 = 1/13.$$

Example. In our example with the contingency table given below,

	in favor	against	total
male	15	45	60
female	4	36	40
total	**19**	**81**	**100**

P(in favor | male)=P(in favor and male)/P(male)=(15/100)/(60/100)=0.25

Note. By reduction of the sample space, we computed *P(in favor | male)=15/60=0.25*

<u>Example</u>. In our example with the contingency table given below,

	Job		
Driver License	No	Part-time	Full-time
yes	34	56	22
no	28	23	7

P(part-time job | no license)=P(part-time job and no license)/P(no license)

=(23/170)/((28+23+7)/170)=0.3966

Note. By reduction of the sample space, we computed

P(part-time job | no license)=23/(28+23+7)=0.3966

Example. The probability that a student is a female is 0.6. The probability that a student is a female and works full-time is 0.27. Find the conditional probability that a randomly selected student works full-time, given that the student is a female.

Solution. $P(works|female) = P(works\ and\ female)/P(female) = 0.27/0.6 = 0.45$.

Multiplication Rule

Rule. The joint probability of events A and B may be computed as

$$P(A \cap B) = P(A \mid B) P(B).$$

Proof. Direct consequence of the definition of the conditional probability.

Example. The probability that a doctor correctly diagnoses a disease is 0.7. The conditional probability that a patient starts a law suit given that the doctor makes an incorrect diagnosis is 0.4. Find the probability that a patient starts a law suit.

Solution. $P(\textit{law suit}) = P(\textit{law suit} \mid \textit{misdiagnosed}) \cdot P(\textit{misdiagnosed}) = (0.4)(1-0.7) = (0.4)(0.3) = 0.12.$

Example. A box contains 8 red and 12 white marbles. Two marbles are drawn, one at a time, without replacement. Find the probability that both marbles are red.

Solution. By the multiplication rule,

$P(1^{st}\ red\ and\ 2^{nd}\ red) = P(2^{nd}\ red \mid 1^{st}\ red)P(1^{st}\ red)$.

Now, $P(1^{st}\ red) = 8/20$. Given that the 1st marble is red, there are 7 red marbles remaining in the box out of 19 total marbles, therefore,

$P(2^{nd}\ red \mid 1^{st}\ red) = 7/19$. Thus, $P(1^{st}\ red\ and\ 2^{nd}\ red) = (7/19)(8/20) = 0.147$.

4.3 Independent Versus Dependent Events

Definition. Two events A and B are called **independent** if their joint probability is the product of the respective probabilities, that is, if $P(A \cap B) = P(A)P(B).$

Recall that we had another definition of independence: $P(A|B)=P(A).$

Proposition. These two definitions are equivalent.

Proof. Suppose $P(A \cap B) = P(A)P(B)$. Then by the definition of conditional probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

Suppose now $P(A \mid B) = P(A)$. By the multiplication rule,

$$P(A \cap B) = P(A \mid B)P(B) = P(A)P(B).$$

Example. One card is drawn from a standard deck of cards. Are the events *A= an ace is drawn* and *B= a black card is drawn* independent?

Solution. Recall that previously we concluded that *A* and *B* are independent because $P(A|B)=2/26=1/13=P(A)$. Now using the new definition of independence, we have to check that $P(A \cap B) = P(A)P(B)$.

We have

$$P(A \cap B) = P(black\ ace) = 2/52 = 1/26,$$

$$P(A) = 4/52 = 1/13, \text{ and } P(B) = 26/52 = 1/2.$$

Therefore,

$$P(A \cap B) = 1/26 = (1/13)(1/2) = P(A)P(B),$$

so A and B are independent.

Example. In our example with a contingency table given below, we checked that the events *male* and *against* are **not independent** by showing that

$$\mathrm{P}(male) = 0.6 \quad \text{but} \quad \mathrm{P}(male \mid against) = \frac{45}{81} = 0.56 \neq 0.6,$$

	in favor	against	total
male	15	45	60
female	4	36	40
total	**19**	**81**	**100**

Using the new definition of independence, we confirm that

$P(male\ and\ against) = 45/100 = 0.45 \neq$
$P(male)P(against) = (60/100)(81/100)$

$= 0.486,$

and thus *male* and *against* are **not independent**.

Example. An office building has two fire detectors that work **independently** of each other. The probability that a fire detector will fail to go off during a fire is 0.02. Find the probability that both fire detectors will fail to go off.

Solution. Since both detectors act independently, $P(1^{st} \text{ fails and } 2^{nd} \text{ fails})$ $= P(1^{st} \text{ fails})P(2^{nd} \text{ fails}) = (0.02)(0.02) = 0.0004$.

Proposition. If $A_1, A_2, \ldots, A_n$ are independent events, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2)\ldots.P(A_n).$$

Example. Suppose in the previous example, there are three detectors. Then, the probability that they will all fail is *P*(*1st fails and 2nd fails and 3rd fails*)

=*P*(*1st fails*)*P*(*2nd fails*)P(*3rd fails*) ***(by independence)***

= (0.02)(0.02)(0.02)=0.000008.

Note. Adding an independent detector tremendously decreases the probability of fire not being detected.

Example. Five students, independently of each other, are taking a test. Each of them has a 0.3 probability to fail. Find the probability that they all will pass.

Solution. By independence, *P(all pass)=*
=P(1st passes)P(2nd passes) $\cdots$ *P(5th passes)*

$$= (1-0.3)^5 = (0.7)^5 = 0.168.$$

Example. Three fair coins are flipped independently. The probability of each outcome is properly assigned using independence of the flips. For instance,

$$P(HTH)=P(H)P(T)P(H) \quad \textbf{(by independence)}$$

$$=(1/2)(1/2)(1/2)=1/8$$

That is why all eight outcomes are equally likely, that is,

$$P(HHH)=P(HHT)=\ldots=P(TTT)=1/8.$$

Example. Suppose we have a biased coin with $P(H)$=0.6 and $P(T)$=0.4. We flip the coin independently until the first head or three tails appear. The sample space is $S=\{H, TH, TTH, TTT\}$.

We assign probability to the outcomes as follows:

$P(H)=0.6$, $P(TH)=P(T)P(H)$ **(by independence)**

$=(0.4)(0.6)=0.24$, $P(TTH)=P(T)P(T)P(H)$

$=(0.4)(0.4)(0.6)=0.096$, and $P(TTT)=P(T)P(T)P(T)$

$=(0.4)(0.4)(0.4)=0.064$.

Note. These probabilities must add up to one.

Indeed, 0.6+0.24+0.096+0.064=1.

4.5 Union of Events. Addition Rule

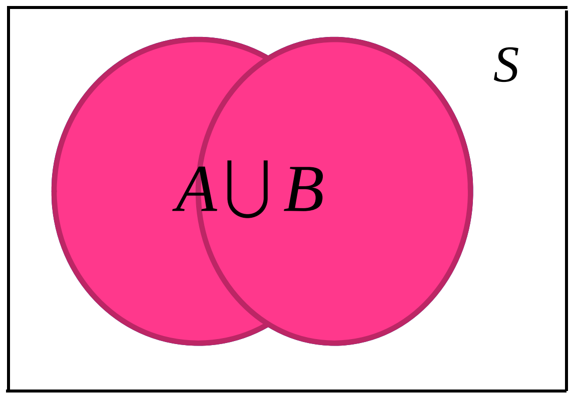
Definition. The **union** of two events A and B is a collection of all outcomes that are either in A or in B or in both.

Notation. $A \cup B$ (read "A union B").

Example. Suppose $A = \{1,3,5\}$, and $B = \{1,3,4,6\}$. Then the union of A and B is

$$A \cup B = \{1,3,4,5,6\}.$$

The union of two events on a Venn diagram looks like this:

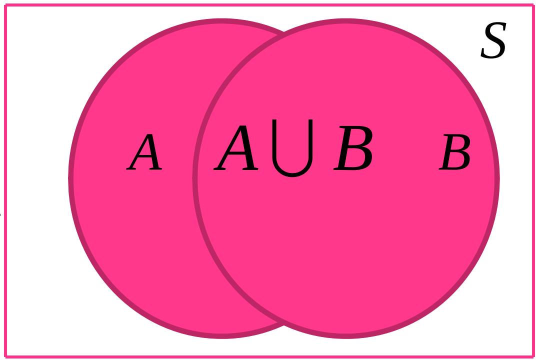


Addition Rule

Rule. For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof. We add $P(A)$ and $P(B)$, but then $P(A \cap B)$ is counted twice, so we have to subtract it once. We visualize this argument by drawing a Venn diagram.



Example. A sample of students and faculty were asked their opinion on some proposal. The data are

	in favor	against	neutral	total
student	90	110	30	230
faculty	45	15	10	70
total	135	125	40	300

Find the probability that a person is a student or is in favor of this proposal.

	in favor	against	neutral	total
student	90	110	30	230
faculty	45	15	10	70
total	135	125	40	300

Solution 1. We can use the definition of the union of two events to write

$$P(student \cup in\ \ favor) = \frac{45+90+110+30}{300} = \frac{275}{300} = 0.92.$$

Solution 2. We can use the addition rule to write

$$P(student \cup in\ \ favor) = P(student) + P(in\ \ favor)$$

$$- P(student \cap in\ \ favor) = \frac{230}{300} + \frac{135}{300} - \frac{90}{300} = \frac{275}{300} = 0.92.$$

Example. In a group of 250 persons, 140 are females, 60 are vegetarian, and 40 are female and vegetarian. Find the probability that a person is male or vegetarian.

Solution. Let $M = male, V = vegetarian$. We have

$$P(M) = 1 - \frac{140}{250} = 0.44,\ P(V) = \frac{60}{250} = 0.24,$$

$$P(M \cap V) = \frac{60-40}{250} = \frac{20}{250} = 0.08.$$

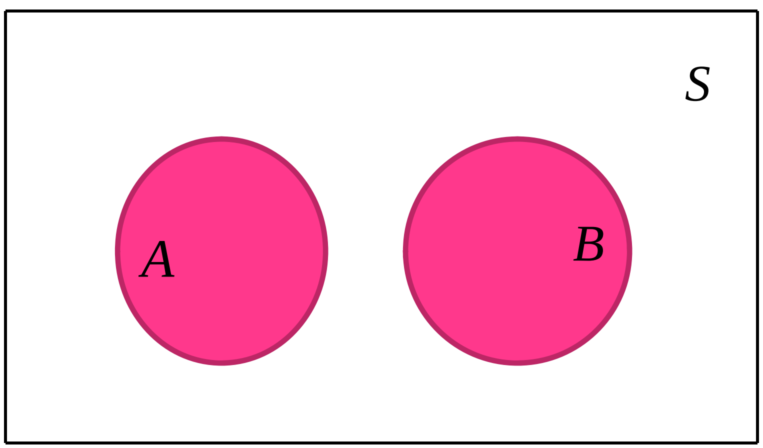
Hence, by the addition rule,

$$P(M \cup V) = P(M) + P(V) - P(M \cap V)$$
$$= 0.44 + 0.24 - 0.08 = 0.6.$$

Two Special Cases of the Addition Rule

1. Disjoint Events: $A \cap B = \varnothing$

   Hence, $P(A \cup B) = P(A) + P(B)$



2. Independent Events: $P(A \cap B) = P(A)P(B)$.

Hence, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.

Example. There are two independent fire engines in a city. The probability that an engine is available when needed is 0.6. What is the probability that at least one engine is available when needed?

Solution. $P(\textit{at least one available}) = P(1^{st} \text{ or } 2^{nd})$

$=0.6+0.6-(0.6)(0.6)= 0.84.$