4.2 Calculating Probability

Definition. **Probability** is a numerical measure of how likely it is that a specific event will occur.

Notation. The probability of an event A is denoted by $P(A)$.

Properties of probability:

(1) $0 \le P(A) \le 1, P(\varnothing) = 0, P(S) = 1.$

(2) If $S = \{E_1, E_2, \dots, E_n, \dots\}$, then

$$P(E_1) + P(E_2) + \dots + P(E_n) + \dots = P(S) = 1$$

Example. Flip a coin once. The sample space is $S=\{H, T\}$. How to assign probabilities to the outcomes?

Probabilities must satisfy:

$0 \leq P(H) \leq 1,\ 0 \leq P(T) \leq 1,$ and $P(H)+P(T)=1.$

For instance, we can assign equal probabilities $P(H)=P(T)=1/2.$

Definition. A coin that is equally likely to fall heads or tails when flipped is called a **fair coin**. Otherwise, the coin is termed a **biased** coin (or **loaded**, or **unfair** coin).

A loaded coin may have, for example, $P(H)=0.8$ and $P(T)=0.2$.

Definition. A **fair die** is equally likely to fall on either of the six sides when tossed. It means that

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6.$$

Probability of Equally Likely Outcomes

Consider n equally likely outcomes $E_1, E_2, \ldots, E_n$. We have $P(E_1)+\ldots+P(E_n)=1$ and also $P(E_1)=P(E_2)=\ldots=P(E_n)$. Therefore, to each simple event, we assign probability $1/n$, that is,

$$P(E_1) = P(E_2)\ldots = P(E_n) = 1/n.$$

Probability of Any Event

In general, to any event $A = \{E_1, E_2, \ldots, E_n\}$, we assign probability

$$P(A) = P(E_1) + P(E_2) + \ldots + P(E_n).$$

Example. A fair coin is flipped twice. Find the probability to see at most one head.

Solution. $S=\{HH, HT, TH, TT\}$. Each outcome is equally likely since the coin is fair, therefore,

$$P(HH)=P(HT)=P(TH)=P(TT)=1/4.$$

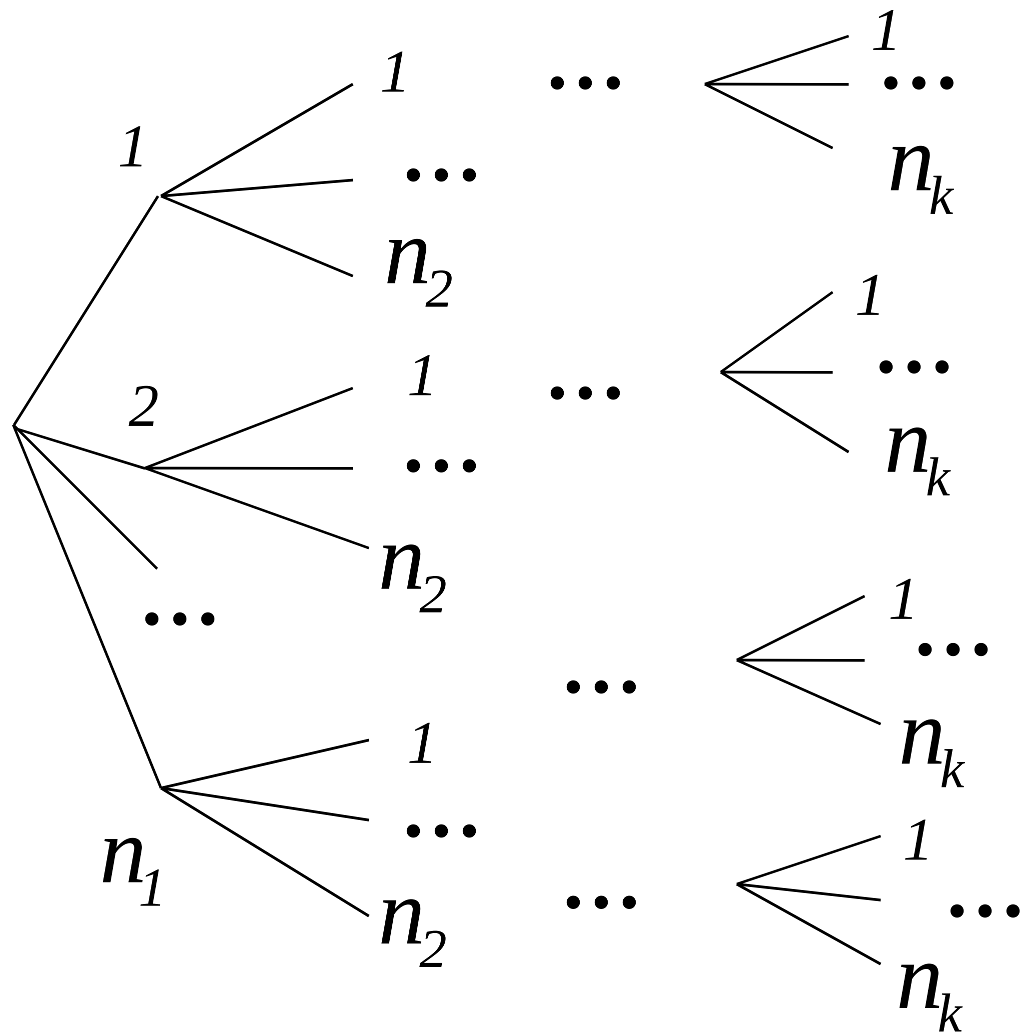
Let A=*at most one head* $= \{HT, TH, TT\}$.

$$P(A)=P(HT)+P(TH)+P(TT)=1/4+1/4+1/4=3/4$$

4.6 Counting Rule

Rule. If an experiment consists of k steps, and the steps result in $n_1, n_2, \ldots, n_k$ outcomes, respectively, then the total number of outcomes for the experiment is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$.

To see this, draw a **tree diagram**.

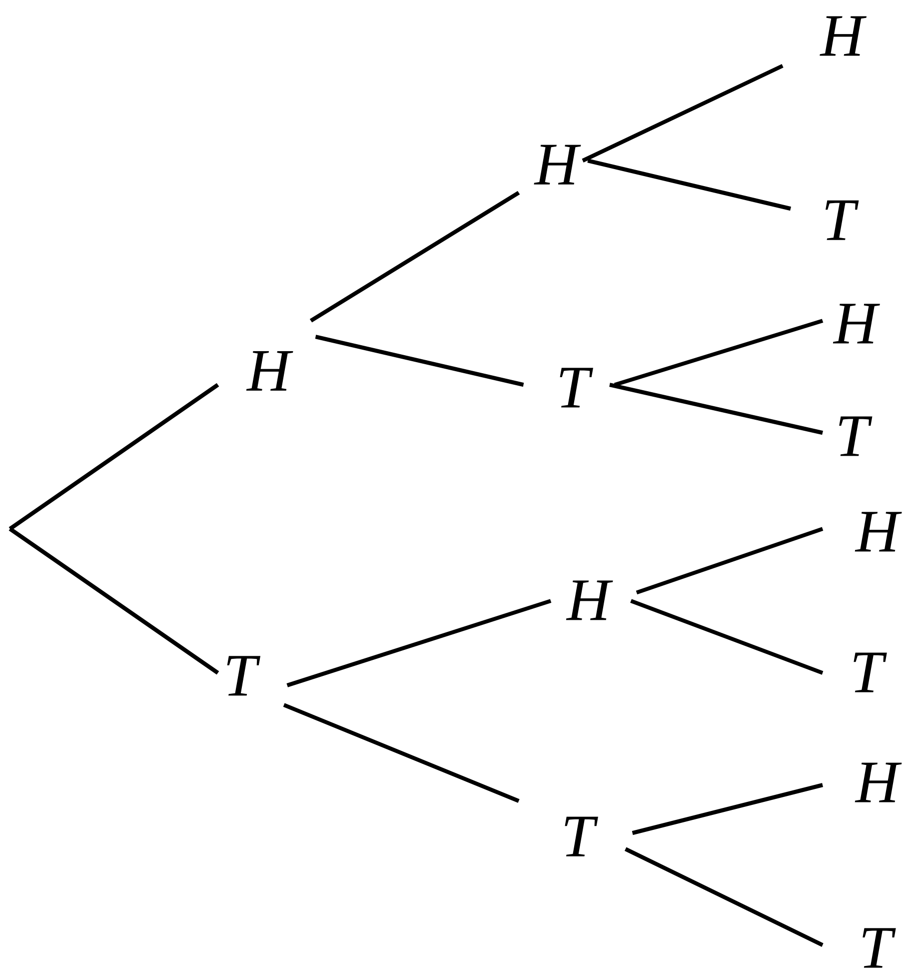


The total number of outcomes is the total number of branches of this tree. It is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$.

Example. A coin is flipped three times. Each flip results in **two** possible outcomes, therefore, the total number of outcomes in the sample space is

$$2 \cdot 2 \cdot 2 = 8.$$

The tree diagram looks like this:



$$S=\{HHH,HHT,HTH,THH,HTT,THT,TTH,TTT\}$$

Example. In a cafeteria there is a choice of 3 appetizers, 5 main courses, 3 desserts, and 4 drinks. How many dinners are possible?

Answer. $3 \cdot 5 \cdot 3 \cdot 4 = 180$.

Example. How many license plates of the form 1ABC234 are possible?

Answer. $10 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$

Example. A quiz consists of five multiple-choice questions with four possible answers each. What is the total number of different ways in which this quiz could be solved?

Answer. $(4)(4)(4)(4)(4)=1024$

4.3 Marginal and Conditional Probabilities

<u>Example</u>. One hundred employees of a company were asked whether they are in favor or against paying high salaries to CEOs of US companies. For each respondent, two characteristics were recorded: *gender* (*male* or *female*) and *opinion* (*in favor* or *against*).

A two-way classification of the responses of these100 employees is summarized in the following table, called **contingency table**.

	in favor	**against**
male	15	45
female	4	36

Marginal Probability

Definition. The **marginal probability** is the probability of a single event without consideration of any other event.

Example. In our example, we can compute the marginal probabilities of the events : $P(male)$, $P(female)$, $P(in\ favor)$, and $P(against)$.

	in favor	against	total
male	15	45	60
female	4	36	40
total	**19**	**81**	**100**

grand total

To compute a marginal probability, take the respective total and divide by the grand total.

$$P(male) = \frac{total\ \#\ of males}{grand\ total} = \frac{60}{100} = 0.6,$$

$$P(female) = \frac{total\ \#\ of\ females}{grand\ total} = \frac{40}{100} = 0.4,$$

$$P(in\ favor) = \frac{total\ \#\ in\ favor}{grand\ total} = \frac{19}{100} = 0.19,$$

$$P(against) = \frac{total\ \#\ against}{grand\ total} = \frac{81}{100} = 0.81$$

Conditional Probability

Definition. The **conditional probability** is the probability that one event will occur given that another event has already occurred.

Notation. If A and B are two events, then the conditional probability of A given B is denoted by $P(A/B)$ and is read “the conditional probability of A given B”.

For contingency tables, the conditional probabilities are computed by the **reduction of the sample space** technique.

Example. In our example, compute *P(in favor | male),* the conditional probability that the respondent is *in favor* given that the respondent is *male.*

We know that the event *male* has occurred. Hence, we reduce the sample space to the *male* row.

	in favor	against	total
male	15	45	60

Now, we look at how many respondents in this row are *in favor*.

$$\mathrm{P}(in\ \ favor\,|\,male) = \frac{\#\,of\ \ males\,in\ \ favor}{total\,\#\,of\ \ males} = \frac{15}{60} = 0.25.$$

<u>Exercise</u>. In our example, compute *P(female | against)*.

<u>Solution</u>. We reduce the sample space to *against* column.

	against
male	45
female	36
total	81

$$\mathrm{P}(\,female\,|\,against) = \frac{\#\,of\;\;females\;against}{total\;\#\;against} = \frac{36}{81} = 0.44.$$

Exercise. A survey of 170 recent high school graduates was conducted, and the data were recorded in a two-way table.

	Job		
Driver's License	No	Part-time	Full-time
yes	34	56	22
no	28	23	7

1. What is the probability that a person randomly chosen from this sample has a driver's license?

Answer. $P(license)=(34+56+22)/170=0.6588$

2. What is the conditional probability that a randomly chosen person has a part-time job, given that he or she has no driver's license?

Answer. $P(\text{part-time job} \mid \text{no license})$

$=23/(28+23+7)=0.3966$

4.3 Mutually Exclusive Events

Definition. Events that cannot occur together are called **mutually exclusive** (or **disjoint**).

Example. A die is rolled. Let *A = an even number is observed = { 2,4,6}, B = an odd number is observed = {1,3,5}, C = a number below 5 is observed = {1,2,3,4}.*

- Are A and B disjoint?

Answer. Yes, they have no outcomes in common.

- Are A and C disjoint?

Answer. No, they have outcomes 2 and 4 in common.

4.3 Independent Versus Dependent Events

Definition. Two events are said to be **independent** if the occurrence of one event doesn't change the probability of the other.

In other words, two events A and B are **independent**, if $P(A|B)=P(A)$.

Definition. If two events are not independent, then they are called **dependent** (or **not independent**). That is, for dependent events A and B,

$$P(A \mid B) \neq P(A).$$

Example. One card is drawn at random from a standard deck of cards. Are the events *A=the drawn card is an ace* and *B=the drawn card is black* independent?

Solution. Among 52 cards there are four aces, so $P(A)=4/52=1/13$.

Suppose now B has happened and a black card was drawn. There are 26 black cards, of which 2 are aces.

Therefore, $P(A|B)=2/26=1/13=P(A)$.

So, A and B are independent.

<u>Example</u>. In our example with a contingency table, are the events *male* and *against* independent?

	in favor	against	total
male	15	45	60
female	4	36	40
total	**19**	**81**	**100**

Solution. $\mathrm{P}(male) = 0.6$ but

$$\mathrm{P}(male \mid against) = \frac{45}{81} = 0.56 \neq 0.6,$$

therefore, the two events are **not independent** (are **dependent**).

Exercise. Can mutually exclusive events A and B be independent?

Answer. No! Because if B has happened, now A cannot happen, so

$$P(A \mid B) = 0 \neq P(A).$$

4.3 Complementary Events

Definition. The **complement** of an event A is the event that consists of all outcomes that are in the sample space S but not in A.

Notation. $\bar{A}$ is read "*A bar*" or "*A complement*". Other typical notation is A^c.

Rule. Since $P(A)+P(\overline{A})=P(S)=1,$ we have

$$P(\overline{A})=1-P(A).$$

Example. In our example, *male* and f*emale* are two complementary events. *P(male)=0.6*, hence, *P(female)=1 – P(male)=1 – 0.6 = 0.4*.