6.1 Continuous Probability Distribution

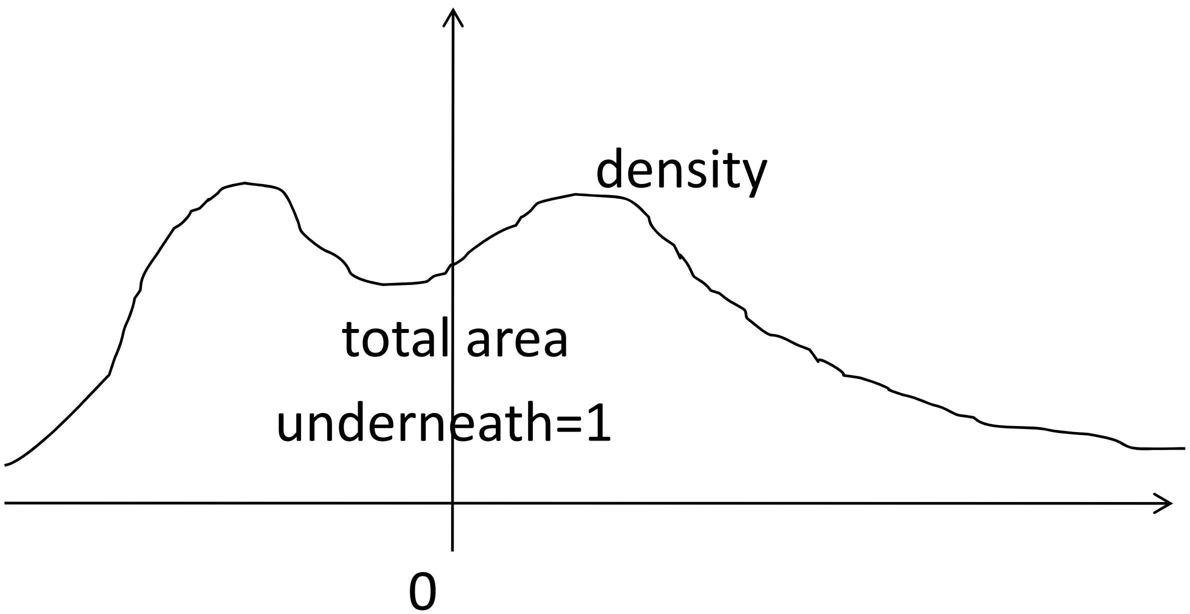
Definition. A **continuous random variable** assumes values in an interval.

Examples. Height, weight, age, commute time, distance to school, length of line in cafeteria, length of song, battery life, rebooting time of a computer.

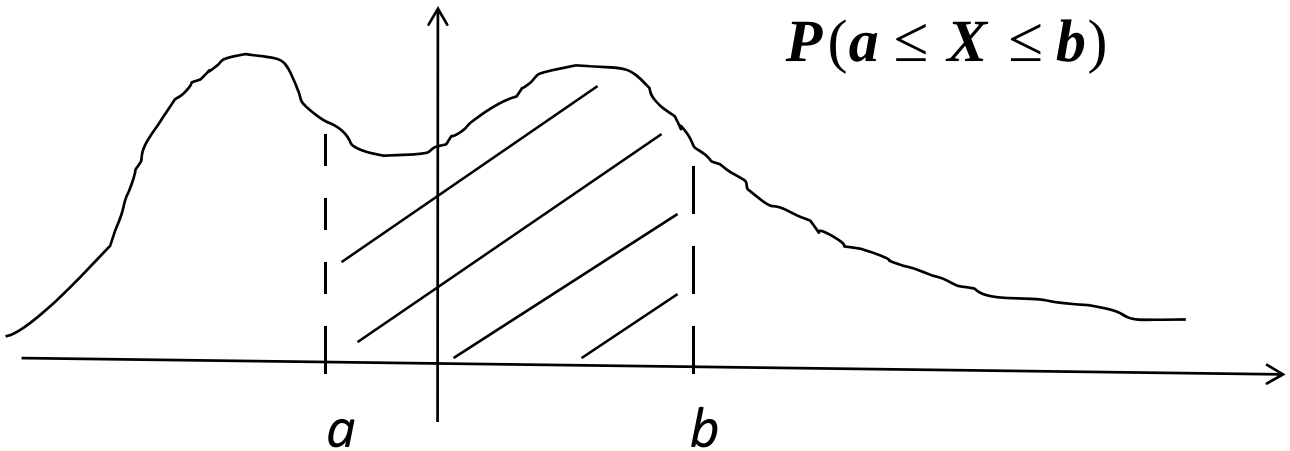
Definition. The **continuous probability distribution** is determined by the **probability density** which is a function that possesses two properties:

- It is everywhere nonnegative
- The total area under this function is equal to one.

Example.



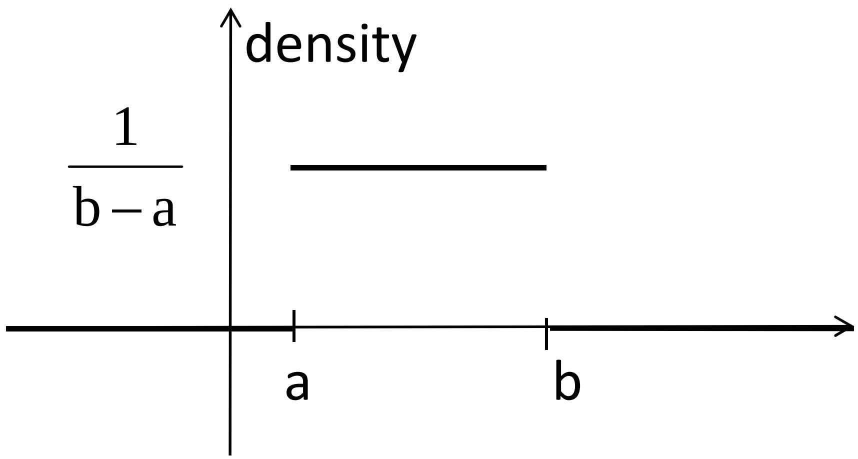
Definition. The probability that X is between two values, say, a and b, is equal to the area under the density curve above the interval $[a,b]$.



Note that the probability that a continuous random variable X is <u>equal</u> to some particular number is always **zero**, $P(X=a)=0$ for any fixed a.

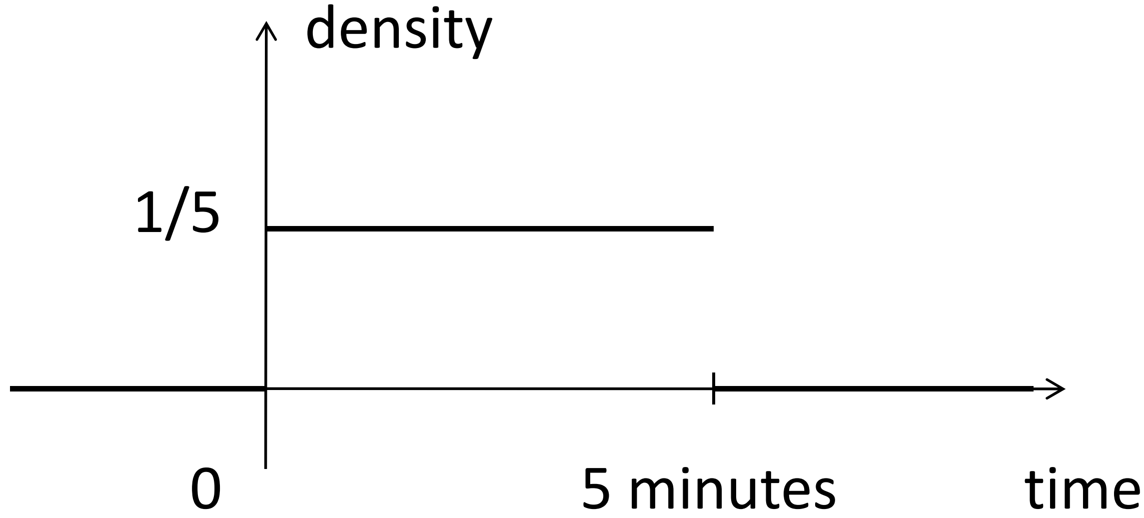
Uniform Distribution

Definition. A random variable X has a **uniform distribution** on an interval $[a, b]$ if the density is constant on this interval and is zero everywhere else. This density looks like this:

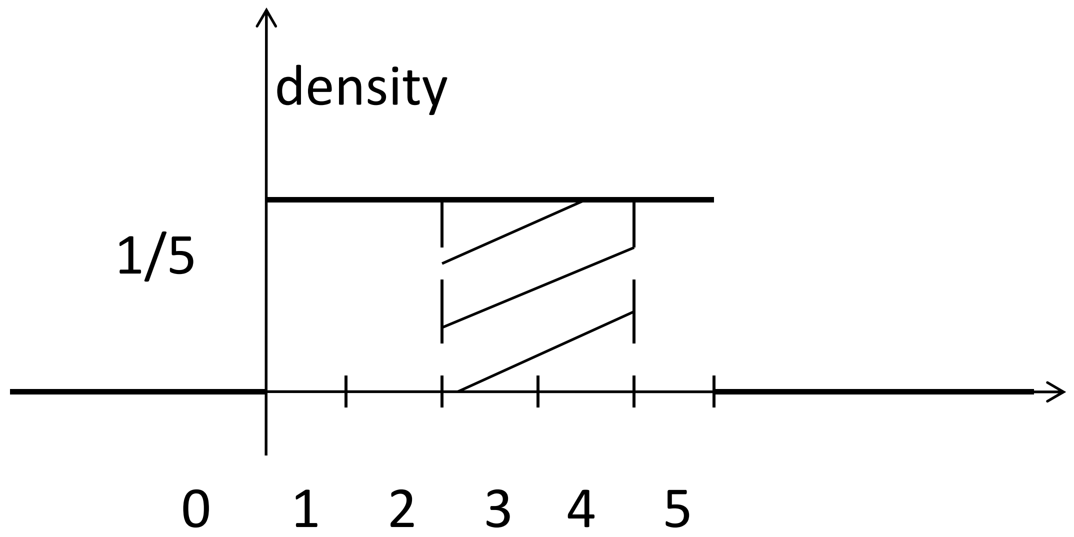


Example. A student comes to a bus stop. The bus will arrive any time within the next five minutes. Find the probability that the student will have to wait between 2 and 4 minutes.

Solution. The wait time T has a continuous distribution with the density that looks like this:



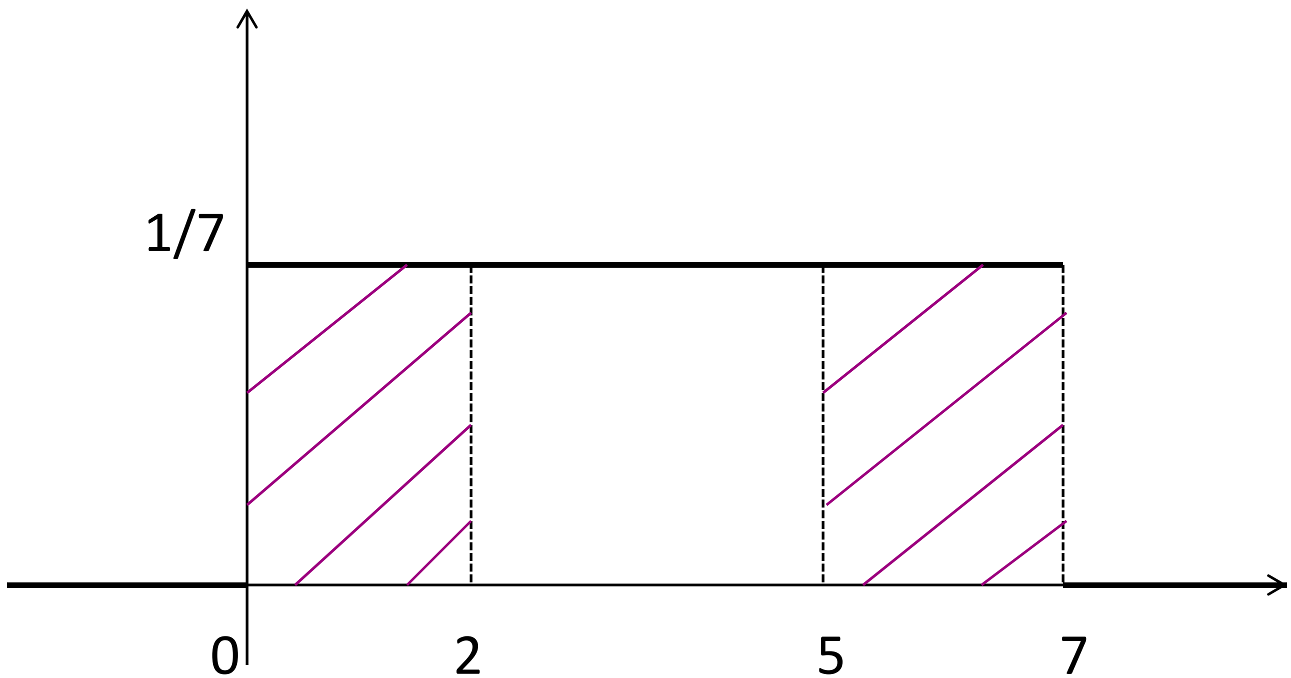
This distribution is uniform on [0,5]. The probability that the wait time is between 2 and 4 is the area under the density curve and above [*2,4*]. So, $P(2 \le \mathrm{T} \le 4) = 2/5$



Example. A 7-foot long stick is cut randomly into two pieces. What is the probability that one piece is less than two feet long?

Solution. Let X be the length of the left piece. Then X is uniformly distributed on the interval [0,7]. We compute

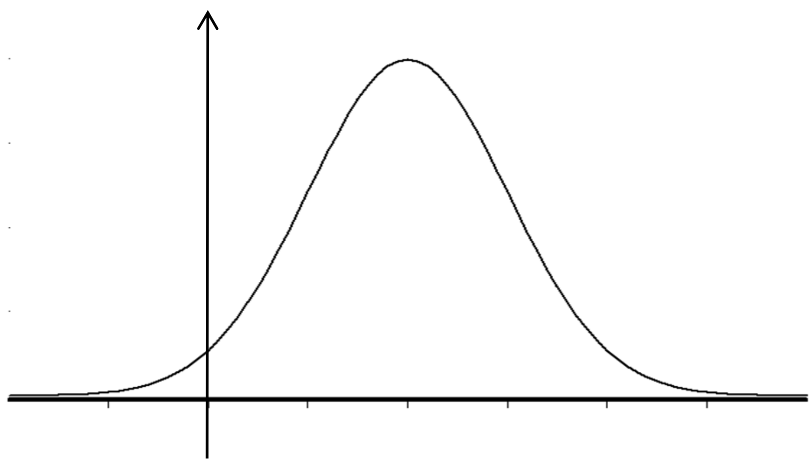
$P(X<2 \text{ or } X>5)=P(X<2)+P(X>5)=2/7 + 2/7=4/7$.



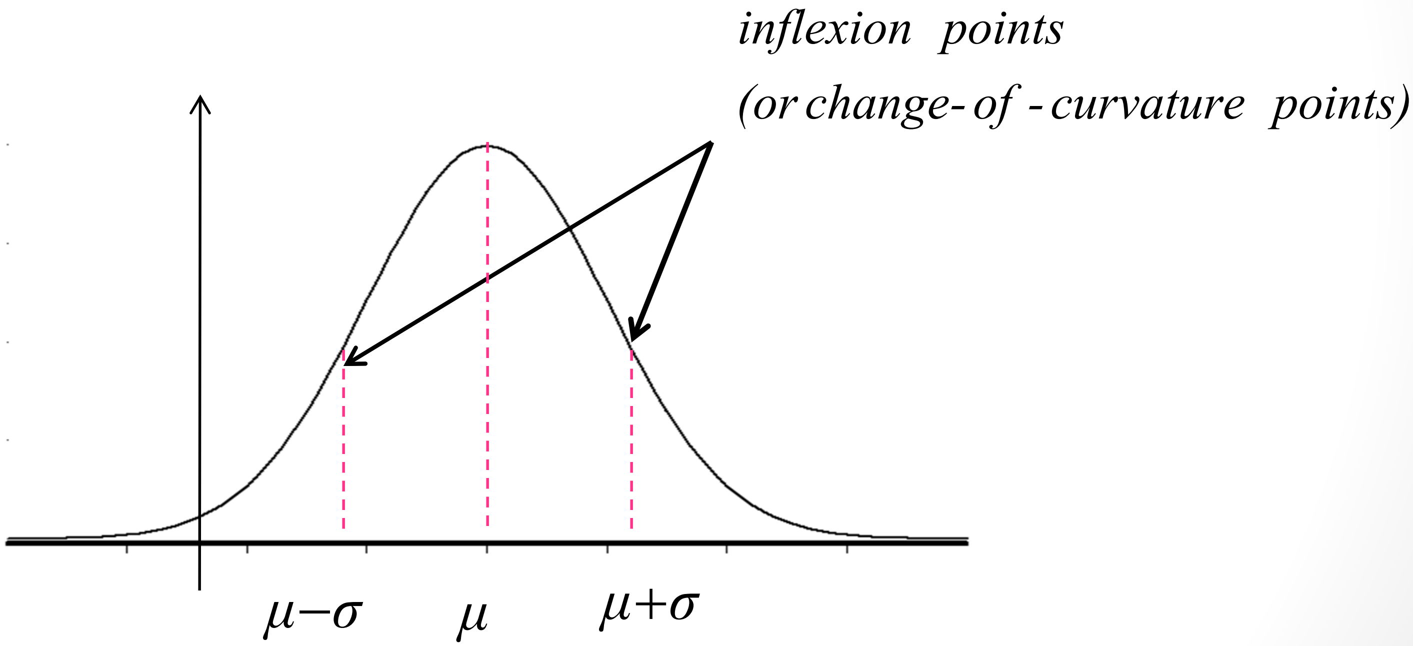
or

6.1 The Normal Distribution

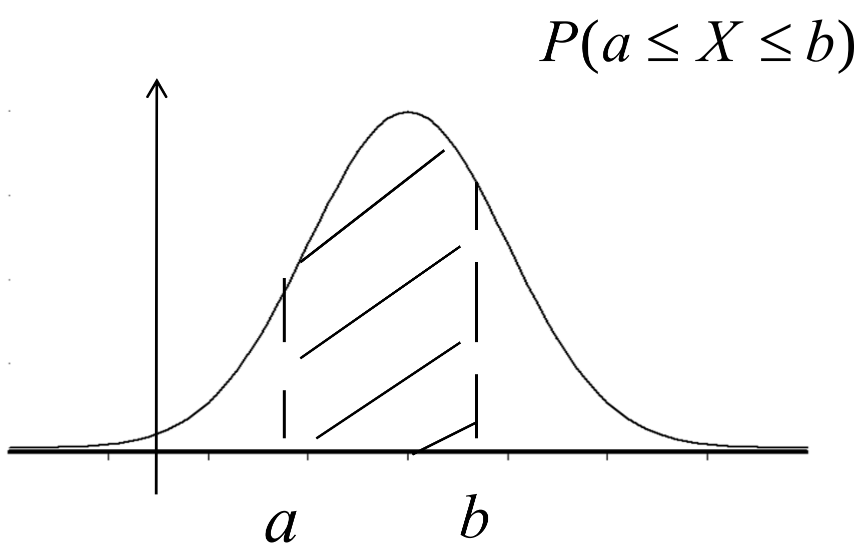
Definition. A continuous random variable defined everywhere on a real line has a **normal distribution** if its density is symmetric and bell-shaped.



If X is a normally distributed random variable with mean μ and standard deviation σ, then the density looks like this:



The probability that a normally distributed random variable *X* lies in an interval [*a, b*] equals to the area under the density curve and above the interval [*a, b*].



Historical Note

Carl Friedrich Gauss (1777 – 1855) was a German mathematician.

He discovered the normal distribution in 1809.

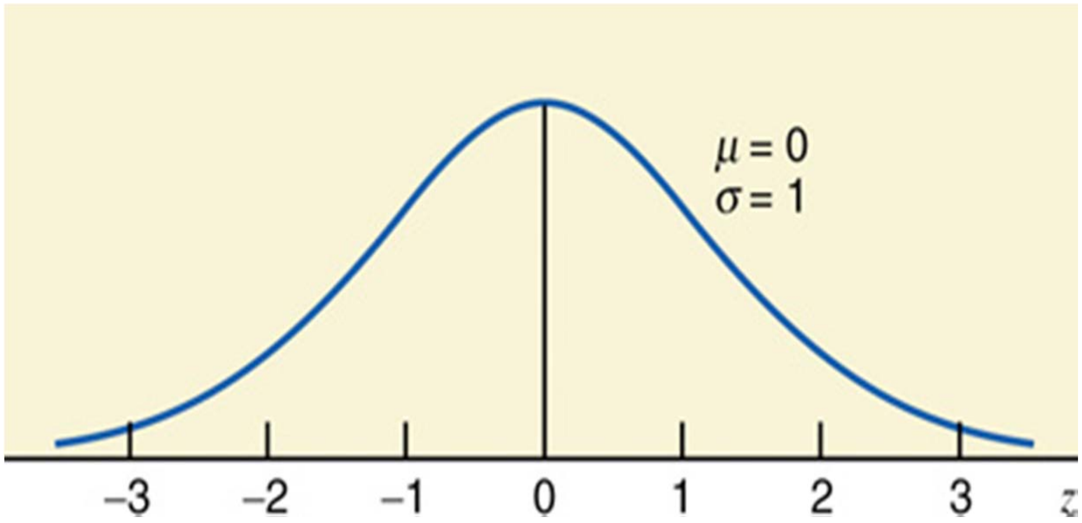
The normal distribution is sometimes called the **Gaussian distribution.**

6.2 Standardizing a Normal Distribution

Definition. The **standard normal distribution** is a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.

Notation. A standard normal random variable is denoted by *z*.

The density of a standard normal distribution looks like this:



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty,$$

$$\pi = 3.14159..., e = 2.71828...$$

How to compute probabilities of a standard normal random variable?

Use the Standard Normal Distribution Table (Table IV in the textbook).

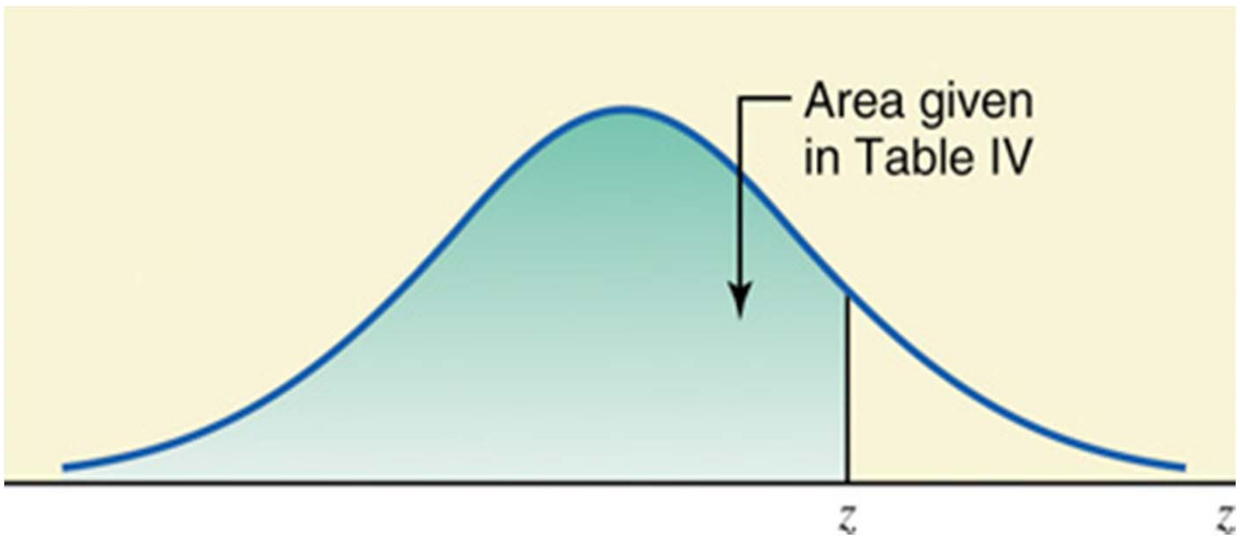


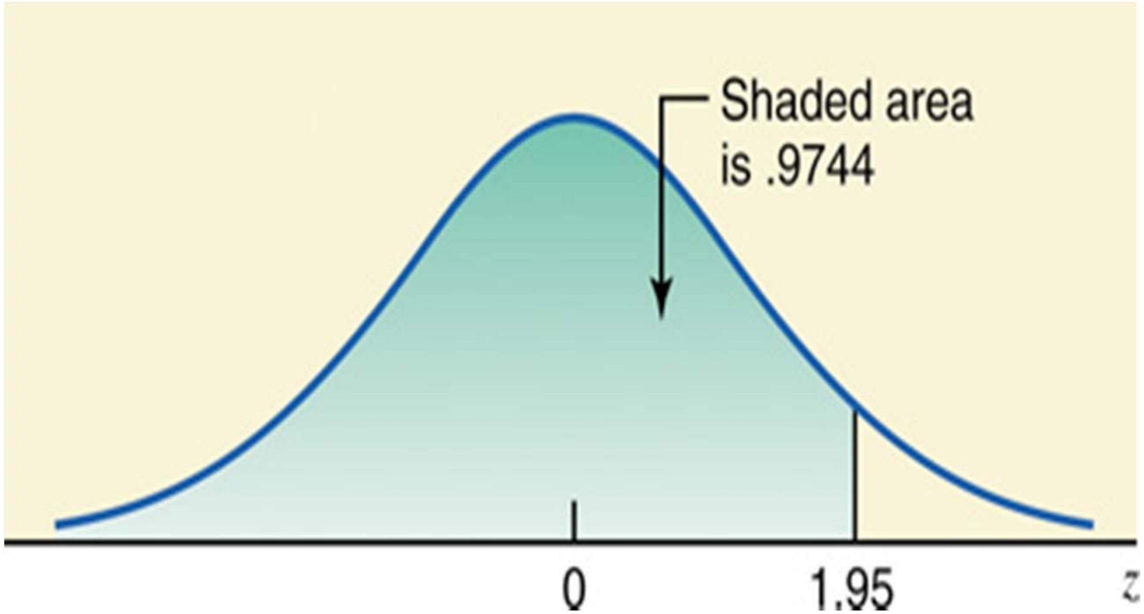
Table 6.2 **Area Under the Standard Normal Curve to the Left of $z = 1.95$**

z	.00	.01	...	.05	...	.09
−3.4	.0003	.0003	...	.0003	...	.0002
−3.3	.0005	.0005	...	.0004	...	.0003
−3.2	.0007	.0007	...	.0006	...	.0005
.	.	.	...	.	...	.
.	.	.	...	.	...	.
.	.	.	...	.	...	.
1.9	.9713	.9719	...	.9744	...	.9767
.	.	.	...	.	...	.
.	.	.	...	.	...	.
.	.	.	...	.	...	.
3.4	.9997	.9997	...	.9997	...	.9998

Required area

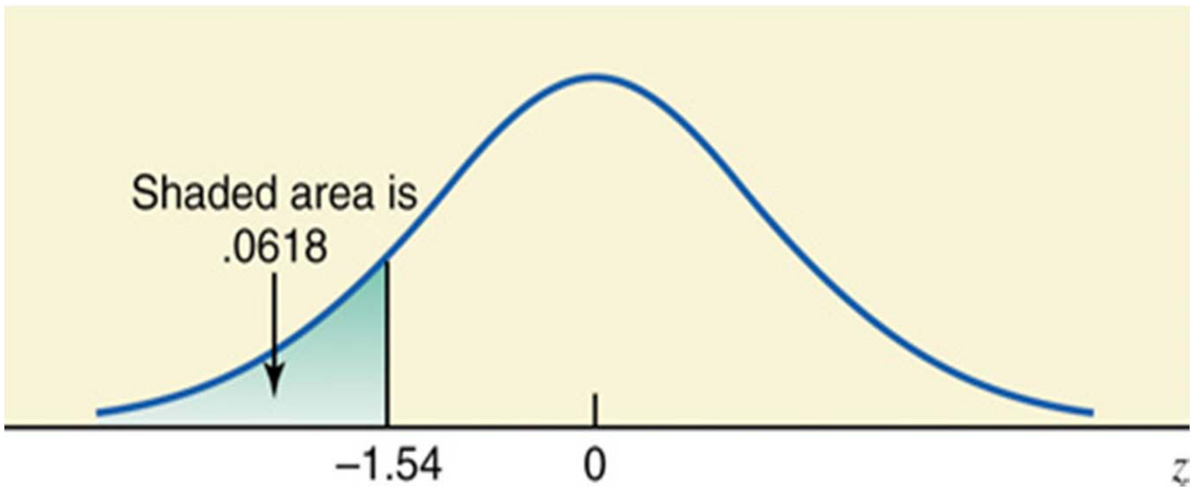
Example.

$$P(z \le 1.95) = 0.9744$$



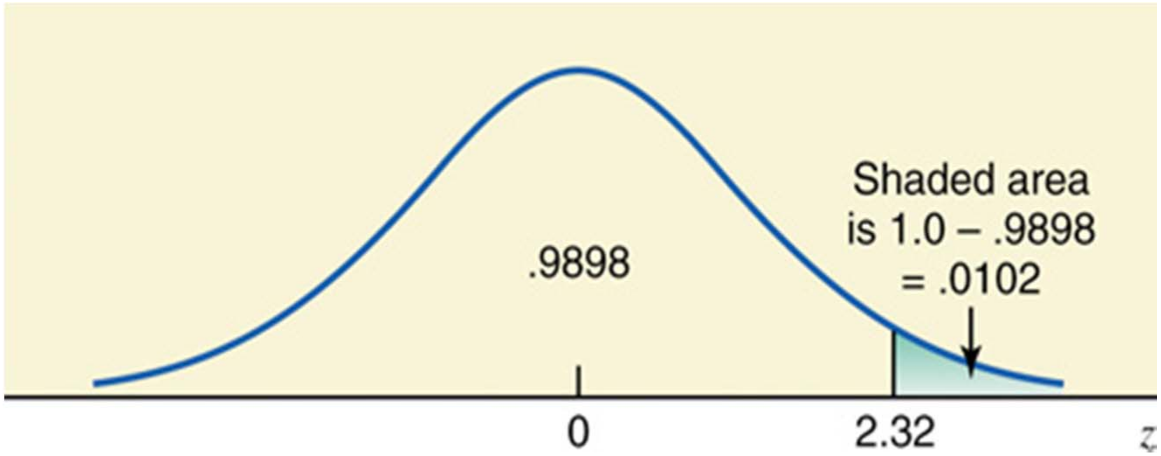
Example.

$$P(z \le -1.54) = 0.0618$$



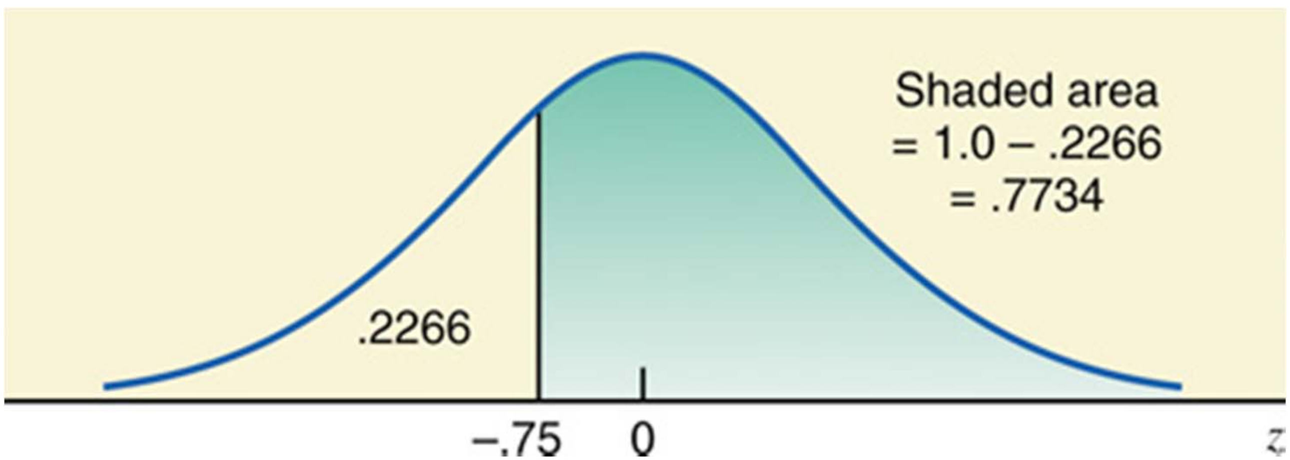
Example.

$$P(z > 2.32) = 1 - P(z \le 2.32) = 1 - 0.9898 = 0.0102$$



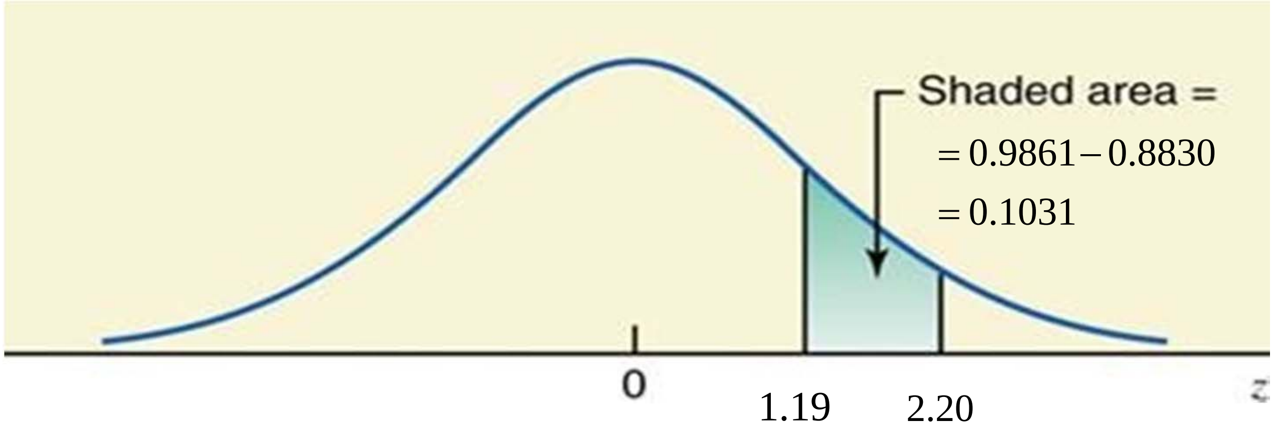
Example.

$$P(z > -0.75) = 1 - P(z \le -0.75) = 1 - 0.2266 = 0.7734$$



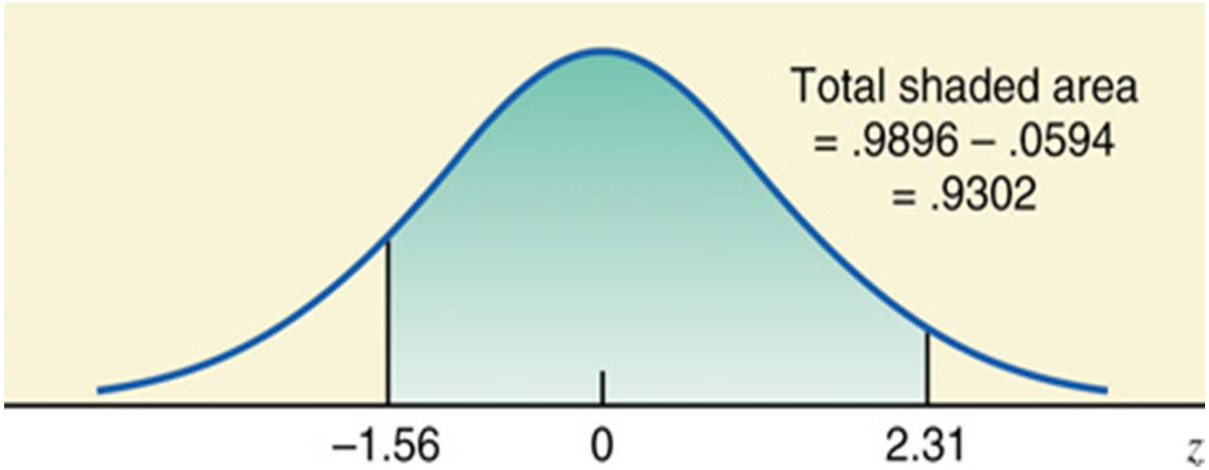
Example.

$$P(1.19 \le z \le 2.20) = P(z \le 2.20) - P(z \le 1.19)$$
$$= 0.9861 - 0.8830 = 0.1031$$



Example.

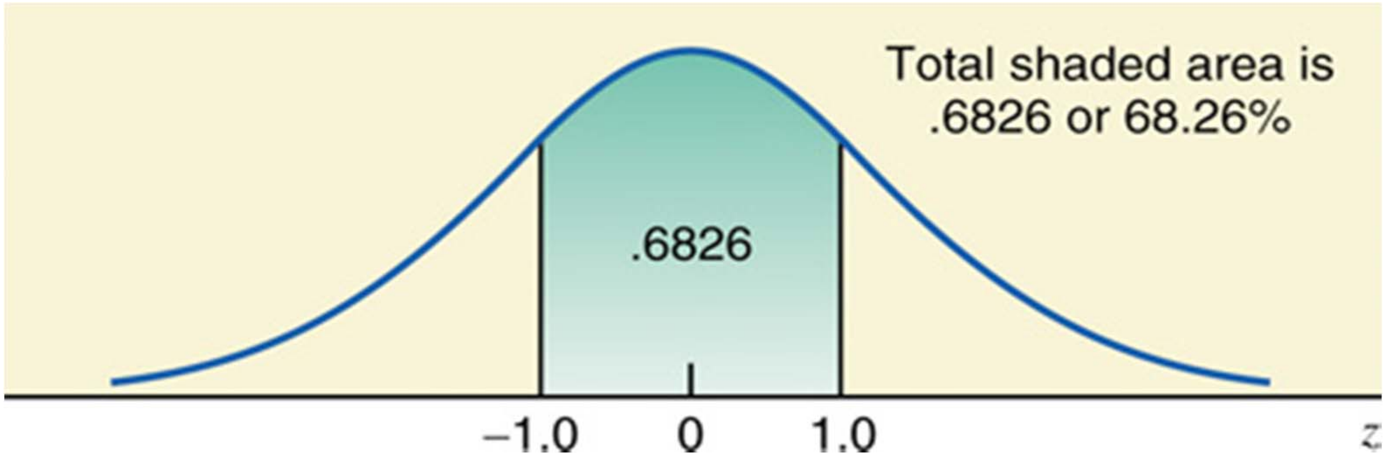
$$P(-1.56 \le z \le 2.31) = P(z \le 2.31) - P(z \le -1.56)$$
$$= 0.9896 - 0.0594 = 0.9302$$



Example.

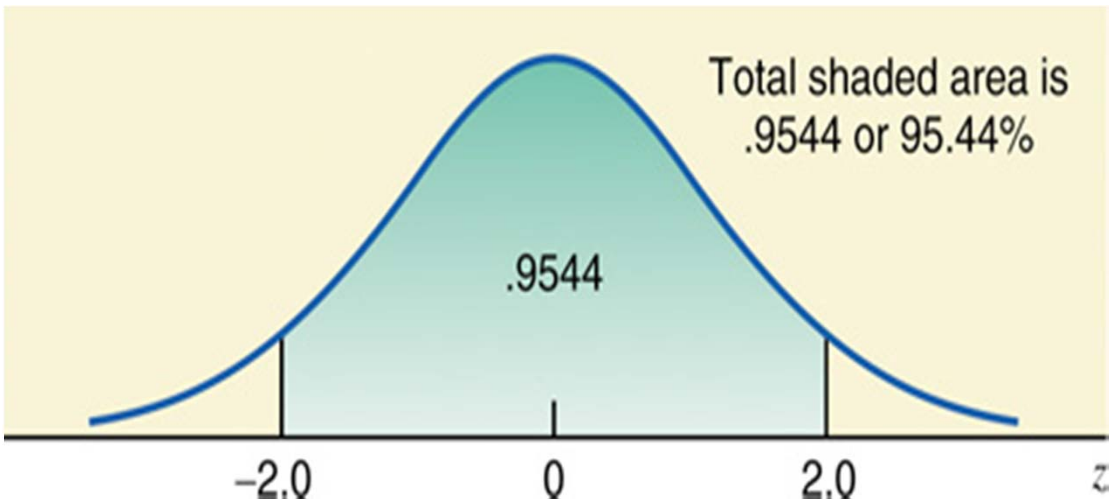
$$P(-1 \le z \le 1) = P(z \le 1) - P(z \le -1)$$

$$= 0.8413 - 0.1587 = 0.6826$$



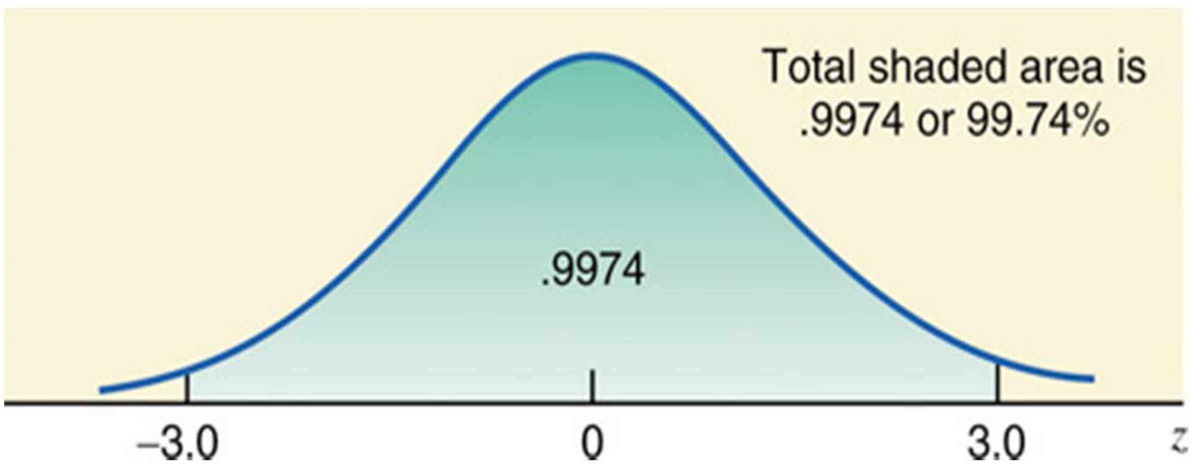
<u>Example</u>.

$$P(-2 \le z \le 2) = P(z \le 2) - P(z \le -2)$$
$$= 0.9772 - 0.0228 = 0.9544$$



Example.

$$P(-3 \le z \le 3) = P(z \le 3) - P(z \le -3)$$
$$= 0.9987 - 0.0013 = 0.9974$$



68-95-99.7% Rule

Rule. About 68% of the observations of a standard normal random variable lie in [-1 , 1], about 95% lie in [-2 , 2], and about 99.7% lie in [-3 , 3].

Note that the Chebyshev theorem states that at least 75% of standard normal random variables lie in [-2, 2], and at least 89% lie in [-3, 3].