6.2 Standardizing a Normal Distribution

Rule. If x is a normally distributed random variable with mean μ and standard deviation σ, then

$$z = \frac{x-\mu}{\sigma}$$

is a standard normal random variable.

Note that

$$x = \mu + z\,\sigma,$$

that is, z tells us how many standard deviations σ above the mean μ (or below, if z is negative) x lies.

Example. Let x be a normally distributed random variable with mean $\mu = 40$ and standard deviation $\sigma = 5$. Find the probability that x assumes a value between 30 and 50.

Solution. Note that the values 30 and 50 are two standard deviations below (respectively, above) the mean. Indeed, $30 = 40 - 2 \cdot 5 = \mu - 2 \cdot \sigma$ and $50 = 40 + 2 \cdot 5 = \mu + 2 \cdot \sigma$

It means that we want to compute the probability that *x* falls within two standard deviations from its mean, which is equivalent to saying that z is inside the interval [-2, 2].

Formally, we write

$$P(30 \le x \le 50) = P\left(\frac{30-40}{5} \le \frac{x-\mu}{\sigma} \le \frac{50-40}{5}\right)$$

$$= P(-2 \le z \le 2) = 0.9544$$

Example. Suppose x is a normal random variable with mean 20 and standard deviation 4.

- We compute

$$P(x \le 16) = P\left(\frac{x-\mu}{\sigma} \le \frac{16-20}{4}\right) = P(z \le -1) = 0.1587$$

- We compute

$$P(x > 26) = P\left(\frac{x-\mu}{\sigma} > \frac{26-20}{4}\right) = P(z > 1.5)$$

$$= 1 - P(z \le 1.5) = 1 - 0.9332 = 0.0668$$

6.3 Applications of the Normal Distribution

Example. The assembly time x of a toy racing car follows a normal distribution with mean 55 minutes and standard deviation 4 minutes. If a worker starts assembling a toy at 4pm, what is the probability that she finishes the job by 5pm?

Solution.

$$P(x \le 60) = P\left(\frac{x-\mu}{\sigma} \le \frac{60-55}{4}\right) = P(z \le 1.25) = 0.8944$$

Example. The life span x of a calculator has a normal distribution with a mean of 54 months and a standard deviation of 8 months. The company guarantees that any calculator that starts malfunctioning within 36 months of the purchase will be replaced. About what percentage of calculators will be replaced?

Solution.

$$P(x \le 36) = P\left(z \le \frac{36-54}{8}\right) = P(z \le -2.25) = 0.0122$$

or about 1.22% .

6.4 Determining z When Area Under Normal Density Curve is Known.

Suppose we know that $P(z \leq z_0) = 0.9251.$

We need to find the number z_0.

Solution. We look inside the normal table and find 0.9251. Now we determine what *z* it corresponds to.

Table 6.4 Finding the z Value When Area Is Known

z	.00	.01	⋯	$\boxed{.04}$	⋯	.09
−3.4	.0003	.0003	⋯		⋯	.0002
−3.3	.0005	.0005	⋯		⋯	.0003
−3.2	.0007	.0007	⋯		⋯	.0005
.	.	.	⋯		⋯	.
.	.	.	⋯		⋯	.
.	.	.	⋯		⋯	.
$\boxed{1.4}$				$\boxed{.9251}$	⋯	⋯
.	.	.	⋯	.	⋯	.
.	.	.	⋯	.	⋯	.
.	.	.	⋯	.	⋯	.
3.4	.9997	.9997	⋯	.9997	⋯	.9998

We locate this value in Table IV of Appendix C

Answer.

$$P(z \leq 1.44) = 0.9251,$$

hence $z_0 = 1.44.$

Example. Scores *x* on a test follow a normal distribution with mean 70 and standard deviation 17. If only the top 15% are given A's, what is the cut-off for an A?

Solution. We need to find x_0 such that $P(x \geq x_0) = 0.15.$ We have

$$P(x < x_0) = 1 - P(x \geq x_0) = 1 - 0.15$$
$$= 0.85 = P\left(z < \frac{x_0 - 70}{17}\right).$$

Hence, $\frac{x_0 - 70}{17} = 1.04$ or $x_0 = 87.68.$

Example. In the example about calculators, what should the warranty period be if the company doesn't want to replace more than 1% of calculators?

Solution. We need to find x_0 such that $P(\mathrm{x} \le \mathrm{x}_0) = 0.01$. We have $0.01 = P\left(z \le \frac{x_0 - 54}{8}\right)$

From here,

$$\frac{x_0 - 54}{8} = -2.33 \quad \text{or} \quad x_0 = 35.36 \quad \text{or} \quad 35 \text{ months.}$$

Example. A random variable *x* has a normal distribution with some unknown mean μ and standard deviation $\sigma = 3$. It is given that $P(x \geq 5) = 0.9066$. We need to find the mean μ.

Solution. We write

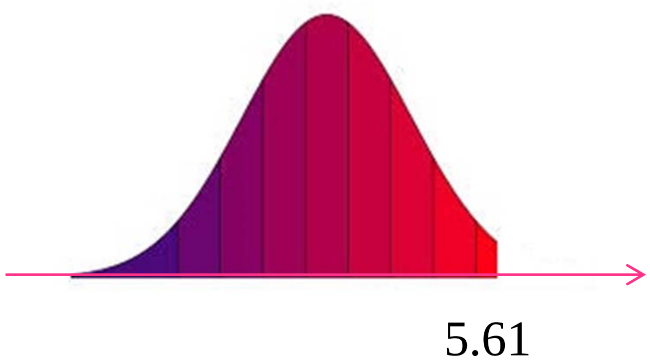
$$P(x \leq 5) = 1 - 0.9066 = 0.0934 = P\left(z \leq \frac{5-\mu}{3}\right).$$

Thus, $\frac{5-\mu}{3} = -1.32$ or $\mu = 8.96.$

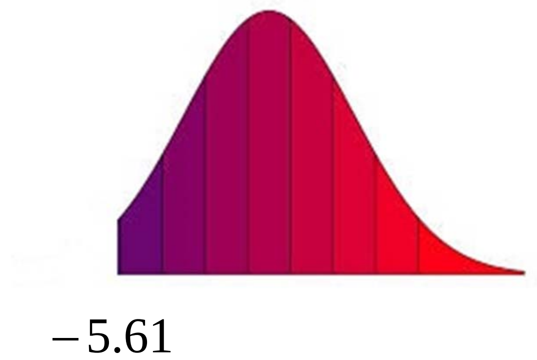
Some important facts

- $P(z = 3.45) = 0$ because the probability of exact equality to a number is equal to zero for any continuous random variable.

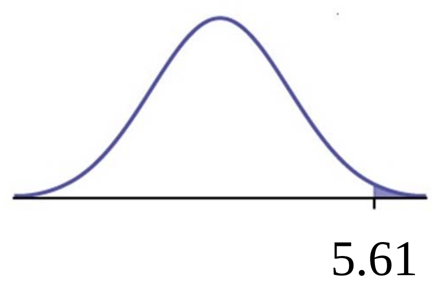
- $P(z \le 5.61) = 1^{-}$ as seen on the graph



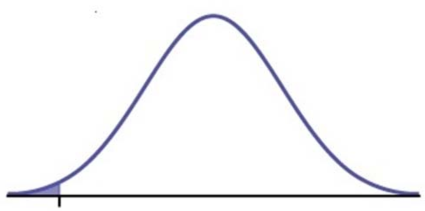
- $P(z \geq -5.61) = 1^{-}$ as seen on the graph



- $P(z \geq 5.61) = 0^{+}$ as seen on the graph

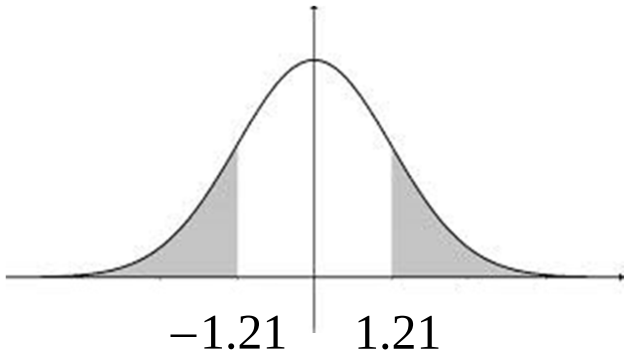


- $P(z \le -5.61) = 0^{+}$ as seen on the graph



- $P(z \ge 1.21) = P(z \le -1.21) = 0.1131$

since the standard normal curve is symmetric around zero.



Alternatively, we could've computed, as before,

$$P(z \geq 1.21) = 1 - P(z \leq 1.21) = 1 - 0.8869 = 0.1131.$$