9.1 Hypothesis Tests: An Introduction

Suppose a sample of size 100 produced the sample mean $\bar{x} = 12.5$. We are Interested in finding out whether the true population mean $\mu > 12$, say.
To answer this question, we perform a **hypothesis test.**

We test the **null hypothesis** $H_0 : \mu = 12$ against the **alternative hypothesis** $H_1 : \mu > 12.$

- Note that what we want to check becomes our alternative hypothesis.
- Technically speaking, the null hypothesis should be $H_0 : \mu \leq 12,$ but for simplicity of presentation we always assume <u>exact equality</u>.

Generally speaking, we may be testing

Hypothesis	**One-tailed test**	**One-tailed test**	**Two-tailed test**
null	=	=	=
alternative	> or $\geq$	< or $\leq$	$\neq$

The decision may be either:

- accept H_0 (equivalently, reject H_1),

or

- reject H_0 (equivalently, accept H_1).

When making a decision about which hypothesis to accept, we may be committing one of two errors:

our decision	true state of nature	
	H_0 is true	H_1 is true
accept H_0	no error	type II error
accept H_1	type I error	no error

Definition. The **probability of type I error** is $\alpha = P(H_0 \text{ is rejected} \mid H_0 \text{ is true})$.

The value α is called the **significance level of the test**.

Recall that we also denoted by α the confidence level of a confidence interval.

CI's and hypotheses testing are closely related.

9.2 Hypotheses Tests About μ When σ is Known

Example. Suppose a sample of size n=100 produced the sample mean $\bar{x} = 12.5,$ and the population standard deviation is known to be $\sigma = 4.$ Is there sufficient evidence to conclude that the true population mean μ is greater than 12?

We are interested in testing $H_0 : \mu = 12$ against $H_1 : \mu > 12$. We assume that H_0 is true. By the CLT, $\bar{x}$ is approximately normally distributed with mean $\mu = 12$ and standard deviation $\frac{\sigma}{\sqrt{n}} = \frac{4}{\sqrt{100}} = 0.4.$

Thus, $z = \frac{\bar{x} - 12}{0.4}$ is approximately a standard normal random variable.

The observed value of z in our example is

$$z = \frac{12.5 - 12}{0.4} = \frac{0.5}{0.4} = 1.25.$$

How unusual is this observation under H_0?

To answers this question, compute

$$P(z \geq 1.25) = P(z \leq -1.25) = 0.1056,$$

so roughly 10.56% of values are above it.

This doesn't seem to be a very unusual observation, and we may assume that the null hypothesis is true, and accept it.

Thus, in our example, we accept $H_0 : \mu = 12$ (or, equivalently, reject $H_1 : \mu > 12$), and conclude that the true population mean is not larger than 12.

Note that even though the observed sample mean $\bar{x} = 12.5$ is larger than 12, we can't statistically conclude that the true population mean is larger than 12. It happens because the standard deviation is large.

Definition. Consider the null hypothesis

$$H_0 : \mu = \mu_0 .$$

The **test statistic** is the observed value of

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} .$$

Example. In our example, $z = 1.25$ is the test statistic.

Definition. Denote by z_0 the observed value of the test statistic. The ***P*-value** for the test is computed as:

- $P(z \geq z_0)$ if $H_1 : \mu \geq \mu_0$ or $\mu > \mu_0$
- $P(z \leq z_0)$ if $H_1 : \mu \leq \mu_0$ or $\mu < \mu_0$
- $P(z \leq -|z_0| \text{ or } z \geq |z_0|) = 2P(z \leq -|z_0|)$

  if $H_1 : \mu \neq \mu_0$

Note. "P" in "*P*-value" stands for "probability".

Example. In our example, $H_1 : \mu > 12$.

Hence,

P-value $= P(z \geq 1.25) = P(z \leq -1.25) = 0.1056.$

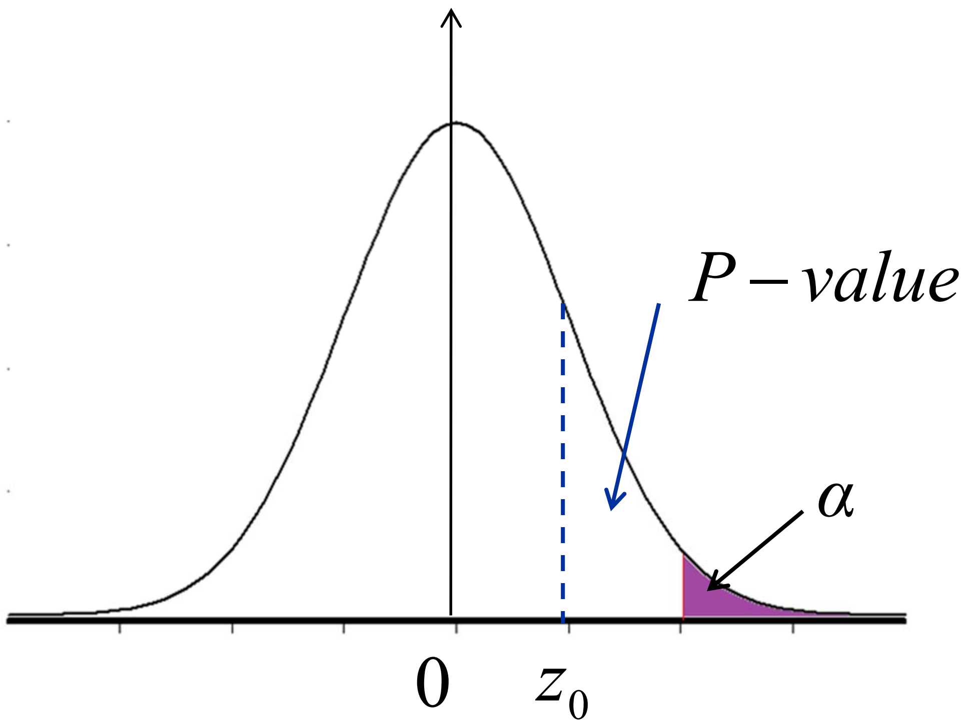
Suppose we carry out the test at a significance level α.

Rule.

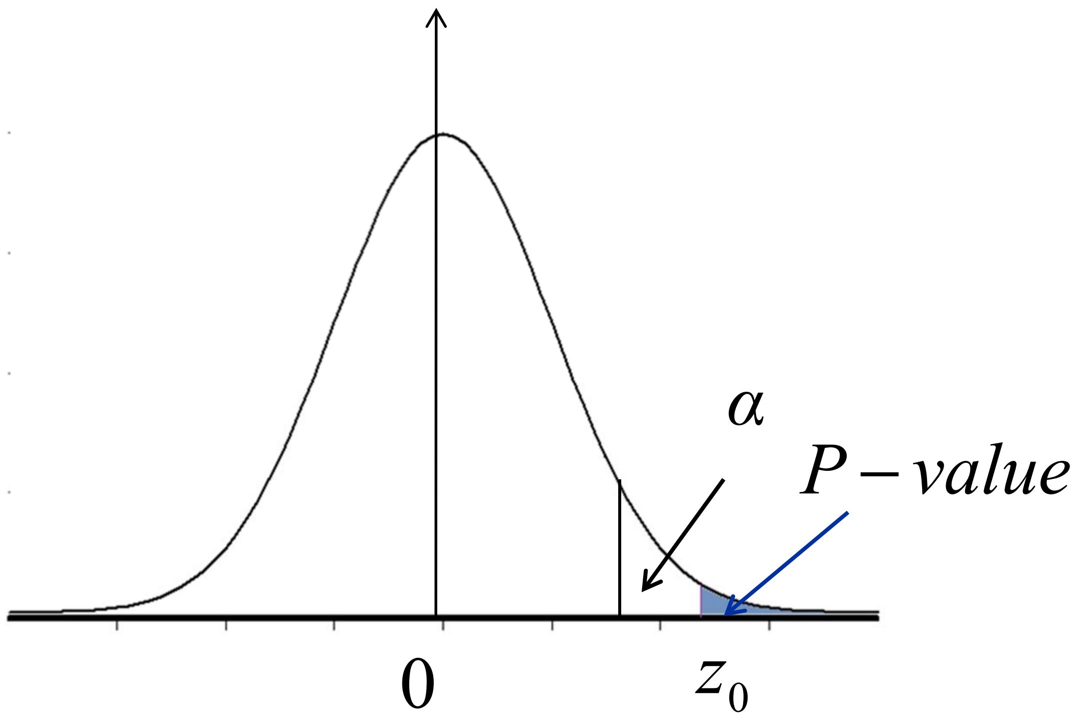
- If P-value > α, then we **accept** H_0.
- If P-value < α, then we **reject** H_0.

Note. If α is not given, then assume $\alpha = 0.05$.

If P-value > α, the observation <u>is not unusual</u> under H_0, so we accept it.



If P-value < α, the observation <u>is unusual</u> under H_0, so we reject it.



Example. In our example, suppose we are testing at the 5% significance level, that is, $\alpha = 0.05$.

P-value=0.1056 > 0.05, therefore, we accept H_0, and conclude that the true population mean doesn't exceed 12.

How to conduct test of hypotheses

- Write null and alternative hypotheses
- Compute test statistic
- Compute *P*-value
- Compare *P*-value to significance level
- State decision and draw conclusion

Example. A company suspects that a 20-ounce coke bottles actually weigh less than that, and conducts a test of hypotheses based on a random sample of size 50 which produced a sample mean of 19.6 ounces. The population standard deviation is known to be 1 ounce.

Carry out the test at significance level of 0.05, and draw conclusion.

Solution. We test $H_0 : \mu = 20$ against $H_1 : \mu < 20.$ The test statistic is

$$z = \frac{19.6 - 20}{1/\sqrt{50}} = -2.83.$$

P-value = $P(z \le -2.83) = 0.0023 < 0.05 = \alpha,$ therefore, we reject H_0, and conclude that the true mean bottle contents is less than 20 ounces.

Example. The mean flight delay was 15 minutes before the merger of two companies. The CEO wants to find out whether the mean flight delay has changed since the merger. A random sample of size 81 flights is drawn, and the sample average delay time of 14 minutes is recorded. The standard deviation is known to be 5 minutes. Test the hypothesis and draw conclusion.

Solution. We test $H_0 : \mu = 15$ against $H_1 : \mu \neq 15.$ The test statistic is

$$z = \frac{14-15}{5/\sqrt{81}} = -1.8.$$

P-value = $2P(z \leq -1.8) = 2(0.0359)$

$$= 0.0718 > 0.05 = \alpha,$$

therefore, we accept H_0, and conclude that the mean time of flight delays hasn't changed with the merger.

Exercise. A manufacturer claims that the mean fat content of a certain grade of steakburger doesn't exceed 20%. A sample of 36 steakburgers produced the mean fat content of 18.3%. Assuming that the standard deviation of fat content is 3%, carry out an appropriate hypothesis test, at the 1% level of significance, in order to advise the consumer group as to the validity of the manufacturer's claim.

Solution. We test $H_0 : \mu = 20$ against $H_1 : \mu \leq 20$. The test statistic is

$$z = \frac{18.3 - 20}{3/\sqrt{36}} = -3.4.$$

P-value = $P(z \leq -3.4) = 0.0003 < 0.01 = \alpha,$ therefore, we reject H_0, and conclude that the claim is true and that the mean fat content doesn't exceed 20%.