4.6 Factorials, Combinations, and Permutations

Definition. The **factorial** of a number *n*, denoted by *n!* (read "*n factorial*") is the product of all integers from *n* to *1*, that is, $n! = (n)(n-1)(n-2)\cdot \ldots \cdot(2)(1)$

By definition, 0! =1.

Examples.

$0! = 1,$

$1! = 1,$

$2! = (2)(1) = 2,$

$3! = (3)(2)(1) = 6,$

$4! = (4)(3)(2)(1) = 24,$

$5! = (5)(4)(3)(2)(1) = 120$

$6! = 720.$

A Useful Identity

$$n! = n \cdot (n-1)!$$

Proof.

$$n! = n \cdot [(n-1)(n-2) \cdot \ldots \cdot (2)(1)] = n \cdot (n-1)!$$

Examples.

3!=(3)(2!)=(3)(2)=6,

4! = (4)(3!)=(4)(6)=24,

5! = (5)(4!)=(5)(24)=120.

Examples.

$$(5-1)! = 4! = 24$$

$$\frac{5!}{(5-1)!} = \frac{(5)(4!)}{4!} = 5$$

$$(3-2)! = 1! = 1$$

$$(4-4)! = 0! = 1$$

$$\frac{5!}{(5-2)!} = \frac{(5)(4)(3!)}{3!} = 20$$

$$\frac{5!}{4!} = \frac{(5)(4!)}{4!} = 5$$

$$\frac{5!}{3!} = \frac{(5)(4)(3!)}{3!} = (5)(4) = 20$$

$$\frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{(6)(5)(4!)}{2!4!} = \frac{(6)(5)}{2} = (3)(5) = 15$$

Definition. A **combination** is an unordered arrangement of objects.

Definition. A **permutation** is an ordered arrangement of objects.

Example. Suppose we have three objects named A, B, and C. There is one combination: ABC, but six permutations: ABC, ACB, BAC, BCA, CAB, CBA.

Example. Suppose we have three objects: A, B, and C, and we randomly choose two.

There are three possible combinations:

AB, AC, BC,

and six possible permutations:

AB, BA, AC, CA, BC, CB.

Definition. The **total number of combinations** when x objects are chosen from n objects is

$$ {}_nC_x = \frac{n!}{x!(n-x)!} $$

Notation. The symbol ${}_nC_x$ is read "*n choose x*". A more common notation is $\binom{n}{x}$.

Example. The number of ways to choose 2 objects from 3 objects is the number of combinations

$$ {}_3C_2 = \frac{3!}{2!(3-2)!} = \frac{3!}{2!1!} = \frac{6}{2} = 3. $$

<u>Definition</u>. The **total number of permutations** when *x* objects are chosen from *n* objects is

$$ {}_{n}P_{x} = \frac{n!}{(n-x)!} $$

<u>Example</u>. In our example, *n=3, x=2*, therefore, $ {}_{3}P_{2} = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 6 $

Example. A committee of 3 has to be chosen from 20 candidates. How many committees are possible?

Solution. We are interested in the number of unordered arrangements (combinations) of 3 people chosen from among 20 people. We compute

$$ {}_{20}C_{3} = \binom{20}{3} = \frac{20!}{3!(20-3)!} = \frac{20!}{3!17!} $$

$$ = \frac{(20)(19)(18)}{(3)(2)} = (10)(19)(6) = 1140 $$

Example. A club has six members willing to be nominated for office. How many different "president/vice-resident/treasurer" slates are possible?

Solution. We are interested in the number of ordered arrangements (permutations) of 3 people chosen from among 6 people. We compute

$$ {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = (6)(5)(4) = 120 $$

Example. If in a group of 7 people, everyone shakes hands with everyone else, how many handshakes take place?

Solution. We are interested in the unordered arrangement of 2 people (for a handshake) from among 7 people. We compute

$$ {}_7C_2 = \frac{7!}{2!(7-2)!} = \frac{7!}{2!5!} = \frac{(7)(6)}{2} = 21 $$

Example. In how many ways can 5 people sit in a row?

Solution. We are interested in an ordered arrangement of 5 people chosen from among 5 people. We compute

$$ {}_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 120 $$

In general, n objects can be ordered in $n!$ ways.

Proof of the formula $_nP_x = \frac{n!}{(n-x)!}$

Permuting *x* objects chosen from among *n* objects is equivalent to placing *n* objects into *x* slots.

n objects	*n* - *1* objects		*n* - *x* + *1* objects
slot 1	slot 2	...	slot x

In slot 1 we can place either of the *n* objects; in slot 2, either of the remaining *n*-1 objects; and so on; in slot *x*, either of the remaining *n*-*x*+1 objects. By the counting rule, the total number of arrangements is

$$_nP_x = n \cdot (n-1)(n-2)\cdot\cdots\cdot(n-x+1) = \frac{n!}{(n-x)!}$$

Proof of formula $_nC_x = \frac{n!}{x!(n-x)!}$

We just proved that the number of permutations of x objects chosen from among n objects is $_nP_x = \frac{n!}{(n-x)!}$

These are <u>ordered</u> arrangements of x objects. We showed that ordering can be done in $x!$ ways.

Thus, we can choose x objects from among n objects in $_nC_x$ ways, and then order them in $x!$ ways, so the relation holds: $_nC_x \cdot x! = {}_nP_x$

From here,

$$_nC_x = \frac{_nP_x}{x!} = \frac{n!}{x!(n-x)!}$$

Exercise. In how many ways can 4 of 8 books be arranged on a shelf?

Solution.

$$ {}_8P_4 = \frac{8!}{(8-4)!} = \frac{8!}{4!} = (8)(7)(6)(5) = 1680. $$

Exercise. (a) An art appreciation class is asked to rank the paintings 1 through 7. How many different rankings are possible?

Solution. $${}_7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 7! = 5040.$$

(b) If the students are asked to rank only the top three, how many rankings are possible?

Solution. $${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = (7)(6)(5) = 210.$$

Exercise. Ten children divide themselves into team A and team B of five each. How many different divisions are possible?

Solution. The number of ways to choose five children to be on team A is

$$ {}_{10}C_{5} = \frac{10!}{5!\,(10-5)!} = \frac{10!}{5!\,5!} = \frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)} $$

$$ = (3)(2)(7)(6) = 252 . $$

The other five children will be on team B.

Exercise. How many 4-card hands are possible?

Solution. Since the order of cards doesn't matter, the number of different 4-card hands is

$$ {}_{52}C_{4} = \frac{52!}{4!\,(52-4)!} = \frac{52!}{4!\,48!} = \frac{(52)(51)(50)(49)}{(4)(3)(2)} $$

$$ = (13)(17)(25)(49) = 270{,}725. $$

Exercise. A police department consists of ten officers. Five officers patrol the streets, two work at the station, and three are on reserve. How many different divisions are possible?

Solution. From ten officers, we need to choose five officers to patrol the streets. From the remaining five officers we need to choose two to work at the station. and the other three officers will be on reserve.

Thus, the number of divisions is computed as

$$
{}_{10}C_5 \cdot {}_5C_2 = \frac{10!}{5!\,(10-5)!}\,\frac{5!}{2!\,(5-2)!} =
$$

$$
\frac{10!\,5!}{5!\,5!\,2!\,3!} = \frac{10!}{5!\,2!\,3!} = \frac{(10)(9)(8)(7)(6)}{(2)(3)(2)}
$$

$$
= (5)(3)(4)(7)(6) = 2{,}520.
$$