5.4 The Binomial Probability Distribution

Definition. A **binomial experiment** is a random experiment that satisfies the following four conditions:

- There are *n* identical trials.
- The trials are independent, that is, the outcome of a trial doesn't affect the outcomes of the other trials.
- Each trial has two outcomes (often termed **success** and **failure**)
- The probability of success *p* is constant, that is, it doesn't change from trial to trial.

Example. Is flipping a fair coin 10 times a binomial experiment?

Answer. Yes.

- There is a predetermined number of flips ($n=10$).
- The flips are independent (assumed).
- Each trial results in either heads or tails.
- The probability of a success is $p=1/2$ (a fair coin) and is constant from trial to trial.

Example. The following are not binomial experiments:

- Flipping a fair coin until the first head appears (number of trials is not predetermined).
- Drawing one marble at a time without replacement from a box containing 8 white and 12 red marbles (probability of drawing a white, say, marble is not constant).
- Rolling a fair die 15 times (the number of outcomes is more than two).

Definition. Let a random variable X be the **number of successes** in a binomial experiment. Then X is called a **binomial random variable,** and it has a **binomial distribution** with parameters n and p, and the probability distribution

$$P(X = x) = {}_nC_x\, p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \; x = 0, 1, \ldots, n.$$

Proof. Consider a sequence of outcomes of n trials, of which x *are* successes and n-x are failures. There are ${}_nC_x$ ways to find x places for the successes, and the other n-x places will contain failures.

Each success has probability p and each failure has probability 1-p, and the trials are independent. Therefore,

$$P(X = x) = {}_nC_x p^x (1-p)^{n-x}$$

A Note on Notation. The probability of failure is sometimes denoted by $q = 1 - p$, hence, the binomial probability may be written as

$$P(X = x) = {}_{n}C_{x}\, p^{x} q^{n-x}$$

<u>Example</u>. A fair coin is flipped 10 times. The flips are independent.

- Find the probability that exactly 6 heads appear.

<u>Solution</u>. Let *X* be the number of heads in the sequence of 10 flips. Then *X* has a binomial distribution with parameters *n=10*, and *p=0.5*. Therefore,

$$P(X=6) = {}_{10}C_6 (0.5)^6 (1-0.5)^{10-6} = \frac{10!}{6!(10-6)!}(0.5)^{10}$$

$$= \frac{10\cdot \overset{3}{\cancel{9}}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot 3\cdot \cancel{2}}(0.5)^{10} = 210\,(0.5)^{10} = 0.2051$$

- Find the probability that <u>at most one head</u> appears.

<u>Solution</u>.

$$P(X=0)+P(X=1)={}_{10}C_0(0.5)^0(0.5)^{10-0}+{}_{10}C_1(0.5)^1(0.5)^{10-1}$$

$$=\frac{10!}{0!(10-0)!}(0.5)^{10}+\frac{10!}{1!(10-1)!}(0.5)^{10}$$

$$=(1+10)(0.5)^{10}=0.0107$$

<u>Example</u>. On a multiple-choice exam with three possible answers for each of the five questions, what is the probability that a student would get four or more correctly just by guessing? Let *Y* be the number of correct guesses on the exam. Then *Y* has a binomial distribution with parameters *n=5*, and *p=1/3.* We compute

$$P(Y \geq 4) = P(Y = 4) + P(Y = 5) = {}_5C_4(1/3)^4(2/3)^{5-4}$$

$$+ {}_5C_5(1/3)^5(2/3)^{5-5} = \frac{5!}{4!(5-4)!}(1/3)^4(2/3)$$

$$+ \frac{5!}{5!(5-5)!}(1/3)^5 = (5)(1/3)^4(2/3) + (1/3)^5 = 0.0453$$

Definition. The **mean** of a **binomial random variable** X is $E(X) = np$.

Definition. The **variance** of a **binomial random variable** X is

$$\sigma^2 = np(1-p) = npq.$$

Definition. The **standard deviation** of a **binomial random variable** X is

$$\sigma = \sqrt{np(1-p)} = \sqrt{npq}.$$

Example. In our previous example,

- How many questions does the student expect to guess correctly?

Solution. $E(Y) = (5)(1/3) = 1.67.$

- What is the standard deviation of the number of correct guesses?

Solution.

$$\sigma = \sqrt{(5)(1/3)(1-1/3)} = \sqrt{10/9} = 1.0541.$$

Example. Screws produced by a company are defective with probability 0.2, independently of each other. A package of 12 screws is bought.

- What is the probability that at least one screw is defective?

Solution. Let *X* be the number of defective screws in the package. Then *X* is binomial with *n=12*, and *p=0.2*.

$$P(X \geq 1) = 1 - P(X = 0) = 1 - {}_{12}C_0 (0.2)^0 (1-0.2)^{12}$$

$$= 1 - (0.8)^{12} = 0.9313$$

- How many screws in the package we expect to be defective?

Solution. $E(X) = np = (12)(0.2) = 2.4$ defectives

- Find the standard deviation of the number of defectives in the package.

Solution.

$$\sigma = \sqrt{np(1-p)} = \sqrt{(12)(0.2)(1-0.2)} = 1.3856 \text{ defectives}$$

Example. A fair die is rolled four times.

- What is the probability that we obtain fewer than two 6's?

Solution. Let "rolling a 6" be a success, and let *X* be the number of successes in this experiment. Then X has a binomial distribution with *n=4* and *p=1/6*. We compute

$$P(X<2)=P(X=0)+P(X=1)={}_4C_0(1/6)^0(1-1/6)^{4-0}$$
$$+{}_4C_1(1/6)^1(1-1/6)^{4-1}=(5/6)^4+(4)(1/6)(5/6)^3=0.8681.$$

- What is the expected number of 6's in the four rolls of a fair die?

<u>Solution</u> . $E(X) = np = (4)(1/6) = 0.67$ 6's

- What is the standard deviation of the number of 6's?

<u>Solution</u>. $\sigma = \sqrt{np(1-p)} = \sqrt{(4)(1/6)(1-1/6)} = 0.7454$ 6's

Example. A boy is lost in the woods. A search party is organized. Each member of the search party has probability 0.1 of finding the boy, independently of all others.

- How large should the search party be, so that the probability of finding the boy is at least 0.9999?

Solution. Let the search party be of size n.

Then P(finding the boy)=1-P(not finding the boy)$=1-(1-0.1)^n=1-(0.9)^n$.

For $n=87$, $1-(0.9)^n = 0.999896$.

For $n=88$, $1-(0.9)^n = 0.999906$.

Thus, the answer is that the search group should consist of a minimum of 88 people.

- Assuming the search party has 88 people, what is the average number of people who will find the boy?

Solution. $(88)(0.1) = 8.8$ people

- What is the standard deviation of the number of people who will find the boy?

Solution. $\sqrt{(88)(0.1)(1-0.1)} = 2.8$ people

Example. A bus-tour agency sold 12 tickets for a bus with 10 seats. Each person is a no-show with probability 0.2, independently of the others.

- Find the probability that all people who show up for the tour will be accommodated.

Solution. Let X be the number of people who show up. Then X has a binomial distribution with $n=12$ and $p=0.8$, and P(all are accommodated)=

$$P(X \leq 10) = 1 - P(X = 11) - P(X = 12)$$

$$= 1 - (12)(0.8)^{11}(0.2) - (0.8)^{12} = 0.7251 .$$

- How many people does the bus-tour agency expect will show up?

Solution. $E(X) = np = (12)(0.8) = 9.6$ people

- What is the standard deviation of the number of people who show up?

Solution.

$$\sigma = \sqrt{np(1-p)} = \sqrt{(12)(0.8)(1-0.8)} = 1.39 \quad \text{people}$$