MATH 114 Finite Mathematics

Textbook: *"Finite Mathematics: An Applied Approach"* by M. Sullivan and A. Mizrahi, John Wiley & Sons, 2004 (9th edition).

1.1. Rectangular Coordinates. Lines.

Definition. The rectangular (or Cartesian) coordinate system consists of two lines, one horizontal (called the x-axis) and the other vertical (the y-axis) that intersect at the origin O.

Definition. The plane formed by the axes is called the xy-plane.
Definition. Any point P in the xy-plane is specified by the coordinates (x, y).
Example. Plot points $(2, -1)$, $(3, 2)$, etc.

Definition. A linear equation is the relation between x and y of the form $Ax + By = C$ for some constants A, B, and C such that A and B are not both zero.
Definition. The graph of the linear equation is a line.
Example. $A = 1$, $B = 2$, $C = -3$; $A = 0$, $B = 1$, $C = 3$; $A = -2$, $B = 0$, $C = 5$.

Definition. The y-intercept $(0, b)$ of a line is the point at which the graph of the line crosses the y-axis. The x-intercept $(a, 0)$ is where the graph crosses the x-axis.

Definition. The slope m of a line is the rate of change of y with respect to x. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two distinct points and $x_1 \neq x_2$, then $m = \frac{y_2 - y_1}{x_2 - x_1}$.
Example. 45 degree line.

Different Forms of an Equation of a Line.

Name	Given	Equation
Point-slope form	Point (x_1, y_1), slope m	$y - y_1 = m(x - x_1)$
Slope-intercept form	Slope m, intercept b	$y = m\,x + b$
Horizontal line	y-intercept b, $m = 0$	$y = b$
Vertical line	x-intercept a, m=infinity	$x = a$
Line between two points	Points (x_1, y_1) and (x_2, y_2) If $x_1 = x_2$, the line is vertical If $x_1 \neq x_2$, then $m = \frac{y_2 - y_1}{x_2 - x_1}$	 $x = x_1$ $y - y_1 = m(x - x_1)$

Example. Find the equation of the line that
(a) passes through points $(1, 0)$ and $(0, 1)$.
(b) passes through the point $(2, 1)$ and is horizontal.
(c) passes through the point $(2, 1)$ and is vertical.

(d) intersects the y-axis at the point $(0, -2)$ and has the slope $m = -1$.
(e) passes through the points $(1, 1)$ and $(1, 2)$.

Example. A gas company calculates a customer's monthly gas bill as \$7.50 plus \$1.50 for every gas unit used. Write the linear equation that describes this relation.

Example. Jane got a regular job and started adding \$9 per week to her savings account. At the end of 11 weeks, she has \$315 in savings. Write her savings as a linear function of the number of weeks since she started the job.
$y = 9x + 216$

Example. An electric utility computes the monthly electric bill for residential customers with a linear function of the number of kilowatt hours (kWh) used. One month a customer used 1560 kWh, and the bill was \$118.82. The next month the bill was \$102.26 for 1330 kWh used. Find the equation relating kWh used and the monthly bill.
$m = \frac{16.56}{230} = 0.072,\ y = 0.072\,x + 6.5$

Example. The American Automobile Manufacturers Association estimated that 520,000 passenger cars were exported in 2000. If exports are expected to decrease by 15,000 passenger cars per year, find the equation for this linear trend.

Example. The sporting goods store has a sale on mopeds at \$725 each. Give the revenue function.

1.3. Applications.

Example (Prediction). The U.S. Census Bureau data show that in 2001 the municipal solid waste was 211.4 millions of tons, whereas in 2003 this amount increased to 220.2 millions of tons. Write down the linear model and use it to predict the gross waste in 2005. $y = 4.4x - 8593; 229$ millions of tons

Example (Break-Even Point). Cox's Department store pays \$100 each for CD players. The store's monthly fixed costs are \$1000. The store sells the CD players for \$200 each.
(a) Find the cost function C, the price the store pays for x CD players.
$C = 1000 + 100x$.
(b) Find the revenue function R, the profit the store makes from selling x CD players. $R = 200x$
(c) Find the break-even point, that is, how many CD players must be sold to guarantee no loss and no profit for the store.
$1000 + 100x = 200x \implies x = 10$

2.1. Systems of Linear Equations: Substitution.

Definition. A system of linear equations is a set of two or more equations in two or more unknowns.
Example. The following is a system of two linear equations in two unknowns:

$$\begin{cases} x + 3y = 5 \\ 2x - y = -4 \end{cases}$$

Definition. To solve a system of linear equations means to find all values of the variables that satisfy these equations.
Definition. To find the solution of a system of linear equations by the method of substitution do the following:
(1) use one of the equations to express one of the variables as a function of the other,
(2) substitute this function into the other equation and solve it.
(3) plug the value found into the function in part (1).
Example.

$$\begin{cases} x = 5 - 3y \\ 2x - y = -4 \end{cases} \implies \begin{cases} x = 5 - 3y \\ 2(5 - 3y) - y = -4 \end{cases} \implies \begin{cases} x = 5 - 3y \\ 10 - 6y - y = -4 \end{cases}$$

$$\implies \begin{cases} x = 5 - 3y \\ -7y = -14 \end{cases} \implies \begin{cases} x = 5 - 3y \\ y = 2 \end{cases} \implies \begin{cases} x = -1 \\ y = 2 \end{cases}$$

Example.

$$\begin{cases} -3x + y = 0 \\ x + 5y = 8 \end{cases} \implies \begin{cases} y = 3x \\ 16x = 8 \end{cases} \implies \begin{cases} x = 1/2 \\ y = 3/2 \end{cases}$$

Example. The public library sets aside \$2,200 to buy fiction and reference books. They plan to purchase 20 more fiction than reference books. The cost of each fiction book is \$20, and of reference book is \$40. Find the number of books the library should buy in each category. $x = 50$, $y = 30$

Example. Celia has one hour to spend in a gym where she jogs or ride a bike. Jogging uses 15 calories per minute, and cycling uses up 5 calories. How long should she participate in each of these categories in order to burn 650 calories? $x = 35$, $y = 25$

Example. Andy allocates 42 hours per week to study for math and chem courses he is taking. He plans to spend twice as much time on math as on chemistry. Find the number of hours Andy allocates to each subject. $x = 28$, $y = 14$

1.3. Applications.

Example (Mixture Problem). A gasoline supplier has two large gasohol tanks, one containing 8% alcohol and the other containing 13% alcohol. If 2000 gallons of gasohol with 10% alcohol is needed, how many gallons should be taken from each tank to provide the proper mixture? $x = 1200$, $y = 800$

Example (Investment Problem). Emily bought two stocks, one selling for $30 per share and the other for $20 per share. She invested total of $4000. The dividend from the first stock is $2 per share and from the other stock is $1 per share. Emily expects to receive a total of $220 in dividends from the two stocks. How many shares of each stock did she buy? $x = 40$, $y = 140$

Example (Mixture Problem). A store is filling an order for 50 pounds of a mixture of walnuts and pecans. The mixture will sell for $3.90 per pound. Walnuts normally sell for $4.50 per pound, and pecans for $3 per pound. If the income from selling the mixture should be the same as that from selling the nuts separately, how many pounds of each type of nut should be used in filling the order? $x = 30$, $y = 20$

Example (Investment Problem). Mr. Green has $10,000 to invest. He will invest part of this sum into secure bonds that carry 2% interest, and the rest into risky stocks with 10% interest. Mr. Green plans to receive the interest of 5% of the invested sum. How much should he invest into bonds? $x = 6,250,\ y = 3,750$

2.1. Systems of Linear Equations: Elimination.

Definition. The method of elimination consists of replacing the original system of equations by an equivalent system for which the solution is easily found.

Rules for Obtaining an Equivalent System of Equations.

1. Interchanging the equations.
2. Multiplying each side of an equation by the same nonzero constant.
3. Replacing any equation in the system by the sum of that equation and a nonzero multiple of the other equation.

Example.

$$\begin{cases} 2x + 3y = -1 \\ -4x + 5y = 13 \end{cases} \overset{2r_1}{\Leftrightarrow} \begin{cases} 4x + 6y = -2 \\ -4x + 5y = 13 \end{cases} \overset{r_1+r_2}{\Leftrightarrow} \begin{cases} 2x + 3y = -1 \\ 11y = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x + 3y = -1 \\ y = 1 \end{cases} \Leftrightarrow \begin{cases} x = -2 \\ y = 1 \end{cases}$$

Example.

$$\begin{cases} 2x+3y=5 \\ 3x-5y=-2 \end{cases} \Leftrightarrow \begin{cases} 6x+9y=15 \\ -6x+10y=-4 \end{cases} \Leftrightarrow \begin{cases} 2x+3y=5 \\ 19y=-19 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x+3y=5 \\ y=1 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=1 \end{cases}$$

2.2. Matrix Method.

Definition. A <u>matrix</u> is a rectangular array of numbers (called <u>entries</u>). Am $m \times n$ matrix has m rows and n columns.

Definition. A system of linear equations can be <u>represented</u> in a <u>matrix form</u> (by specifying an <u>augmented matrix</u>).
Example.

$$\begin{cases} 2x+3y=-1 \\ -4x+5y=13 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 3 & \mid & -1 \\ -4 & 5 & \mid & 13 \end{bmatrix}$$

The solution of this system is $\begin{cases} x=-2 \\ y=1 \end{cases}$ which is equivalent to the augmented matrix $\begin{bmatrix} 1 & 0 & \mid & -2 \\ 0 & 1 & \mid & 1 \end{bmatrix}$.
Definition. The <u>matrix method</u> of solving a system of linear equations consists of replacing the original augmented matrix by an equivalent matrix for which the solution of the system is easily found. The replacement is done by means of <u>row operations</u>:
1. Interchanging any two rows.
2. Replacing a row by a nonzero multiple of that row.
3. Replacing a row by the sum of that row and a nonzero multiple of some other row.

Example. $\begin{bmatrix} 2 & 3 & \mid & -1 \\ -4 & 5 & \mid & 13 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 & 6 & \mid & -2 \\ -4 & 5 & \mid & 13 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 3 & \mid & -1 \\ 0 & 11 & \mid & 11 \end{bmatrix}$
$\Leftrightarrow \begin{bmatrix} 1 & 3/2 & \mid & -1/2 \\ 0 & 1 & \mid & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3/2 & \mid & -1/2 \\ 0 & -3/2 & \mid & -3/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & \mid & -2 \\ 0 & 1 & \mid & 1 \end{bmatrix}$
Example. $\begin{bmatrix} 2 & 3 & \mid & 5 \\ 3 & -5 & \mid & -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 6 & 9 & \mid & 15 \\ -6 & 10 & \mid & 4 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 3 & \mid & 5 \\ 0 & 19 & \mid & 19 \end{bmatrix}$
$\Leftrightarrow \begin{bmatrix} 1 & 3/2 & \mid & 5/2 \\ 0 & 1 & \mid & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3/2 & \mid & 5/2 \\ 0 & -3/2 & \mid & -3/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & \mid & 1 \\ 0 & 1 & \mid & 1 \end{bmatrix}$
Example. $\begin{cases} x+3y=5 \\ -2x+6y=8 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 3 & \mid & 5 \\ -2 & 6 & \mid & 8 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 6 & \mid & 10 \\ -2 & 6 & \mid & 8 \end{bmatrix} \Leftrightarrow$
$\begin{bmatrix} 1 & 3 & \mid & 5 \\ 0 & 12 & \mid & 18 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3 & \mid & 5 \\ 0 & 1 & \mid & 3/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 3 & \mid & 5 \\ 0 & -3 & \mid & -9/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & \mid & 1/2 \\ 0 & 1 & \mid & 3/2 \end{bmatrix}$

Example. $\begin{cases} \frac{1}{2}x - y = \frac{1}{6} \\ -\frac{1}{4}x + 3y = \frac{1}{3} \end{cases} \Leftrightarrow \begin{bmatrix} \frac{1}{2} & -1 & | & \frac{1}{6} \\ -\frac{1}{4} & 3 & | & \frac{1}{3} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \frac{1}{2} & -1 & | & \frac{1}{6} \\ -\frac{1}{2} & 6 & | & \frac{2}{3} \end{bmatrix} \Leftrightarrow$

$\begin{bmatrix} \frac{1}{2} & -1 & | & \frac{1}{6} \\ 0 & 5 & | & \frac{5}{6} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \frac{1}{2} & -1 & | & \frac{1}{6} \\ 0 & 1 & | & \frac{1}{6} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \frac{1}{2} & 0 & | & \frac{1}{3} \\ 0 & 1 & | & \frac{1}{6} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & | & \frac{2}{3} \\ 0 & 1 & | & \frac{1}{6} \end{bmatrix}$

2.4. Matrix Algebra.

Arranging Data in a Matrix.

Example. An airline is buying two types of airplanes: type I and type II. Type I planes will seat 30 first-class , 50 tourist-class, and 90 economy-class passengers. Type II planes will seat 50 first-class, 60 tourist-class, and 100 economy-class passengers. Display the data in a 2×3 matrix. $\begin{bmatrix} 30 & 50 & 90 \\ 50 & 60 & 100 \end{bmatrix}$

Definition. Addition of matrices is defined for matrices with the same dimensions (numbers of rows and columns) and is performed entry-wise.

Example. Add (a) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ *impossible*

(b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 0 \\ 1 & 3 & 6 \end{bmatrix}$

Definition. The product of a matrix by a number, called scalar multiplication of a matrix, is performed entry-wise.

Example. $2 \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & 8 \end{bmatrix}$

Example. The costs per year of going to a college are:

	State	*Private*
Tuition fees	4500	16000
Living expenses	5000	9400

Assume the annual inflation rate of 5%.

(a) Find a matrix expression that shows the increase in costs for the second year. $(0.05) \begin{bmatrix} 4500 & 16000 \\ 5000 & 9400 \end{bmatrix} = \begin{bmatrix} 225 & 800 \\ 250 & 470 \end{bmatrix}$

(b) Find the matrix expression for the actual costs for the second year. $\begin{bmatrix} 4500 & 16000 \\ 5000 & 9400 \end{bmatrix} + \begin{bmatrix} 225 & 800 \\ 250 & 470 \end{bmatrix} = \begin{bmatrix} 4725 & 16800 \\ 5250 & 9870 \end{bmatrix}$

2.5. Multiplication of Matrices.

Definition. The product of a $1 \times r$ matrix $\begin{bmatrix} a_1 & a_2 & \dots & a_r \end{bmatrix}$ and a $r \times 1$ matrix $\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_r \end{bmatrix}$ is defined by

$$\begin{bmatrix} a_1 & a_2 & \dots & a_r \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_r \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_r b_r.$$

Definition. Multiplication of two matrices is defined if the number of columns of the first matrix equals to the number of rows of the second matrix. The product of a $m \times r$ and $r \times n$ matrices is a $m \times n$ matrix which entry in row i and column j is the product of the ith row of the first matrix and the jth column of the second one.

Example. Multiply (a) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$ *impossible*

(b) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 23 & 1 \\ 21 & -5 \\ 18 & -2 \end{bmatrix}$

Example. Costs, in dollars, for radio (per minute), newspaper (per column inch), and TV (per minute) ads in two cities are given in the following matrix:

$$\begin{matrix} & \textit{Radio} & \textit{Newspaper} & \textit{TV} \\ \text{City A} & 30 & 20 & 120 \\ \text{City B} & 25 & 18 & 140 \end{matrix}$$

(a) Ads are run five times in each of these media in City A and eight times in each of the media in City B. Find the total amount spent on radio, newspaper, and TV ads. $\begin{bmatrix} 5 & 8 \end{bmatrix} \begin{bmatrix} 30 & 20 & 120 \\ 25 & 18 & 140 \end{bmatrix} = \begin{bmatrix} 350 & 244 & 1720 \end{bmatrix}$

(b) The ads are run 30, 40, and 60 times, respectively, in each of the cities in January, and 20, 12, and 22 times in February. Find the total amount spent in each city in January and February. $\begin{bmatrix} 30 & 20 & 120 \\ 25 & 18 & 140 \end{bmatrix} \begin{bmatrix} 30 & 20 \\ 40 & 12 \\ 60 & 22 \end{bmatrix} = \begin{bmatrix} 8900 & 3480 \\ 9870 & 3796 \end{bmatrix}$

2.6. The Inverse of a Matrix.

Definition. A 2×2 identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In general, $n \times n$ identity matrix has ones on the main diagonal and zeros everywhere else. The identity matrix has the property that $AI = IA = A$.

Definition. The inverse of a square $n \times n$ matrix A is A^{-1} such that $AA^{-1} = A^{-1}A = I$.

Note. Inverses are defined only for square matrices.

Proposition. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example. $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \frac{1}{(1)(5)-(3)(2)} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

Method to Find the Inverse of an $n \times n$ Matrix A.
1. Write down an augmented matrix $\begin{bmatrix} A & | & I \end{bmatrix}$.
2. Perform a sequence of row operations to reduce the A portion of the matrix to identity matrix. Then the matrix found in the I portion is A^{-1}.

Example. Find $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}^{-1}$.

Solution: $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & -2 & 0 & -3 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & -5 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 5/2 & -1/2 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 5/2 & -1/2 \end{array}\right]$.

Solving Systems of Linear Equations Using Matrix Inverses.

Proposition. A system of n linear equations can be written in matrix form as $A \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$ for some $n \times n$ matrix A and some constants $c_1, \dots, c_n$.

Example. $\begin{cases} x + y = 2 \\ y + z = -1 \\ x + 2z = 3 \end{cases} \leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

The solution of this system is $\begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = A^{-1} \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$.

Proof: $AX = C \leftrightarrow A^{-1}AX = A^{-1}C \leftrightarrow IX = A^{-1}C \leftrightarrow X = A^{-1}C$.

Example. Solve $\begin{cases} 0.5x + 0.3y = 1.3 \\ 25x + 7y = 49 \end{cases}$.

Solution: In matrix form this is $\begin{bmatrix} 0.5 & 0.3 \\ 25 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.3 \\ 25 \end{bmatrix}$. The inverse of $\begin{bmatrix} 0.5 & 0.3 \\ 25 & 7 \end{bmatrix}$ is $\begin{bmatrix} -1.75 & 0.075 \\ 6.25 & -0.125 \end{bmatrix}$. The solution to the system is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1.75 & 0.075 \\ 6.25 & -0.125 \end{bmatrix} \begin{bmatrix} 1.3 \\ 25 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 2.0 \end{bmatrix}$.

Example. A theater charges \$4 for children and \$8 for adults. One weekend, 900 people attended the theater, and the admission receipts totaled \$5840.

Find the number of children and adults attending.

Solution: The information can be represented by $\begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 900 \\ 5840 \end{bmatrix}$.

The inverse of $\begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix}$ is $\begin{bmatrix} 2 & -0.25 \\ -1 & 0.25 \end{bmatrix}$. Therefore, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -0.25 \\ -1 & 0.25 \end{bmatrix} \begin{bmatrix} 900 \\ 5840 \end{bmatrix} = \begin{bmatrix} 340 \\ 560 \end{bmatrix}$.

3.1. Systems of Linear Inequalities.

Definition. A <u>linear inequality</u> is an inequality of the form $Ax + By \geq C$ or $Ax + By > C$ or $Ax + By \leq C$ or $Ax + By < C$. The inequalities with signs $>$ or $<$ are called <u>strict</u> inequalities.

Example. $3x - 2y < 4$, $x + y \geq 1$.

Definition. The solution of a linear inequality is a half-plane lying either above or below the line $Ax+By = C$, depending on the sign of the inequality. If the inequality is nonstrict, the line is a part of the solution.

Steps for Graphing a Linear Inequality.

1. Graph the linear equation using a solid line if the inequality is nonstrict, and using dashes if it is strict.
2. Select a test point above (or below) the plotted line.
3. If the point satisfies the inequality, then all points on the same side of the line solve the inequality.

Example. Solve $2x + y < 3$.

Example. Solve $2x \geq y$.

Graphing a System of Linear Inequalities.

Example. $\begin{cases} x + 3y < 6 \\ x - y \leq 2 \\ x \geq 0 \end{cases}$

Definition. All points that satisfy a system of inequalities make up the <u>feasible region</u> of that system.

Definition. A point of intersection of two boundary lines is called a <u>corner point</u> of the feasible region.

To find a corner point, express the equations of the boundary lines as $y = m_1 x + b_1$ and $y = m_2 x + b_2$. The lines intersect at the point (x, y) where x satisfies $m_1 x + b_1 = m_2 x + b_2$ and y satisfies $y = m_1 x + b_1$ or $y = m_2 x + b_2$, that is, $x = (b_2 - b_1)/(m_1 - m_2)$ and $y = (m_1 b_2 - m_2 b_1)/(m_1 - m_2)$.

In our example, corner points are (0,-2), (0,2), and (3,1).

Example. $\begin{cases} x+y\le 5 \\ x+2y\le 8 \\ x\ge 0 \\ y\ge 0 \end{cases}$ $\begin{cases} -2x+y\le 1 \\ x+y\le 4 \\ y\ge 2 \end{cases}$ $\begin{cases} x>1 \\ x\le 4 \\ y\ge 0 \end{cases}$ $\begin{cases} x\ge y \\ x<2y \\ x\le 3 \end{cases}$

3.2. A Geometric Approach to Linear Programming Problems.

Definition. A linear programming problem consists of maximizing (or minimizing) an objective function $z = Ax + By$ under some constraints given as a system of linear inequalities.

Example. An appliance store has the storeroom capacity limited to 50 items. Each washer takes 2 hours to unpack and set up, and each dryer takes 1 hour. There are 80 hours of employee time available. Washers sell for \$300 each, and dryers sell for \$200 each. How many of each should be ordered to maximize the revenue? x=washers, y=dryers, $z = 300x+200y \to$max, $x+y \le 50,\ 2x+y \le 80,\ x \ge 0,\ y \ge 0.$

The Fundamental Theorem of Linear Programming.
The solution to a linear programming problem is located as a corner point of the feasible region.

Steps for Solving a Linear Programming Problem.
1. Specify the objective function and determine whether it should be maximized or minimized.
2. Determine the set of constraints in the form of a system of linear inequalities.
3. Graph the feasible regions and find the coordinates of the corner points.
4. Calculate the value of the objective function in each corner point.
5. Select the maximum (or minimum) value of the objective function.

In our example,

Corner Point	Objective Function
(0,0)	0
(0,50)	10,000
(30, 20)	13,000
(40,0)	12,000

Answer: 30 washers and 20 dryers.

3.3. Applications.

Example. Jack has a casserole and salad dinner. Each serving of casserole contains 250 calories, 2 milligrams of vitamins, and 9 grams of protein. Each serving of salad contains 30 calories, 6 milligrams of vitamins, and 1 gram of protein. Jack wants to consume at least 20 milligrams of vitamins and 12 grams of protein but keep the calories at a minimum. How many servings of each food should he eat?

Corner Point	Objective Function
(0,12)	360
(1,3)	340
(10,0)	2500

Answer: 1 casserole and 3 salads.

Example. A biology department buys mice and rats for experimental purposes. It buys at least three times as many mice as rats. Each mouse costs \$2, each rat costs \$5, and the departmental budget dictates that no more than \$110 be spent on such purchases. If each mouse can be used for three experiments and each rat for two, how many of each should be purchased to maximize the total number of experiments that can be run?

Corner Point	Objective Function
(0,0)	0
(55,0)	165
(30, 10)	110

Answer: 55 mice and no rats.

Example. A company that makes desk lamps and floor lamps has 1200 hours of labor and \$4200 to purchase materials each week. It takes 0.8 hour of labor to make a desk lamp and 1 hour to make a floor lamp. The materials cost \$4 for each desk lamp and \$3 for each floor lamp. The company makes a profit of \$2.65 on each desk lamp and \$3.15 on each floor lamp. How many of each should be made each week to maximize profit?

Corner Point	Objective Function
(0,0)	0
(0,1200)	3780
(375, 900)	3828.75
(1050,0)	2782.5

Answer: 375 desk lamps and 900 floor lamps.

6.1. Sets.

Definition. A collection of objects (elements) is called a set.
Notation. $A = \{a, b, c\}, \ a \in A, \ d \notin A.$
Example. $A = \{0, 1, 2, 3, 4, 5\}, \ 3 \in A, \ 8 \notin A.$
Definition. Two sets are equal if they have the same elements.
Notation. $A = B, \ C \neq D.$
Example. $A = \{n \,|\, n^2 \leq 5, \ n \text{ integer}\}, \ B = \{-2, -1, 0, 1, 2\}, A = B$
$C = \{1, 3, 4\}, \ D = \{2, 3, 5, 6\}, C \neq D.$
Definition. A set A is a subset of B ($A \subseteq B$), if every element of A is also

an element of B.
Definition. A set A is a proper subset of B ($A \subset B$), if A is a subset of B and there is at least one element that is in B but not in A.
Example. $A = \{1, 2, 3\},\ B = \{0, 1, 2, 3, 4, 5\},\ A \subset B$.
Definition. The union of sets A and B ($A \cup B$) is a set consisting of the elements that are either in A or in B or in both.
Example. $A = \{0, 1, 2, 3\},\ B = \{2, 3, 5\},\ A \cup B = \{0, 1, 2, 3, 5\}$.
Definition. The intersection of sets A and B ($A \cap B$) is a set consisting of the elements that are in both A and B.
Example. $A \cap B = \{2, 3\}$.
Definition. The universal set U is the set consisting of all elements under consideration.

Venn Diagram.
Historical Note. John Venn (1834 – 1923) was the English logician who taught moral sciences at Cambridge University.

Draw a Venn diagram to illustrate subsets, unions and intersections.

Definition. A null (or empty) set $\emptyset$ has no elements.
Definition. Two sets A are B are disjoint if they have no elements in common (or $A \cap B = \emptyset$).
Example. $A = \{2, 3, 4\},\ B = \{5, 6, 7\}, A \cap B = \emptyset$.
Definition. The compliment of a set A ($\bar{A}$) consists of all the elements in U that are not in A.
Example. $A = \{2, 3, 4\},\ U = \{0, 1, 2, 3, 4, 5\},\ \bar{A} = \{0, 1, 5\}$.
DeMorgan's Laws. (a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, (b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
Proof: Draw a Venn diagram.

Example. $A = \{1, 2, 3\},\ B = \{2, 3, 5\},\ C = \{1, 3, 5, 7, 8\},\ U = \{0, 1, 2, 3, 5, 7, 8, 9\}$.
Find (a) $A \cup B$,
(b) $A \cap B$,
(c) $B \cup \bar{C}$,
(d) $\overline{A \cap C}$,
(e) $\bar{B}$,
(f) $A \cup (B \cap C)$,
(g) $(C \cap A) \cup \bar{A}$.

6.2. The Number of Elements in a Set.

Notation. The number of elements in a set A is denoted by $c(A)$.
Proposition (Counting or Additive Formula). $c(A \cup B) = c(A) + c(B) - c(A \cup B)$.
Proof: Draw a Venn diagram.
Example. Given $c(A) = 120,\ c(B) = 100,\ c(A \cap B) = 40$. Find $c(A \cup B)$.
Example. In a clothing store, 50 people bought t-shirts, 24 bought pants,

16 bought both. How many people bought either a t-shirt or pants?

Application of the Venn Diagram.
Example. Of 50 students surveyed, 27 owned a laptop, 39 owned a graphical calculator, and 25 owned both. How many students
(a) did not own a calculator?
(b) owned a calculator but did not own a laptop?
(c) owned neither?
(d) owned one or the other but not both?

Example. The SWR Group advertised for applicants for secretarial, clerical, and typist positions. Respondents could apply for one or more of the positions. The responses were as follows:
12 applied for secretary,
10 for clerk,
14 for typist,
7 for secretary and typist,
5 for secretary and clerk,
3 for clerk and typist,
2 for all three,
3 for none of the positions.
(a) What was to total number of respondents?
(b) How many applied for the typist position only?
(c) How many applied for the secretary position but not the clerk position?
(d) How many applied for exactly one position?
(e) How many applied for exactly two positions?
(f) How many applied for at least one of the positions?

Example. In the course of a day, 40 careful bulls entered a particular china shop. Fifteen of them broke a precious vase, 18 of them broke an invaluable plate, 9 broke both a vase and a plate. All those that didn't break a vase, broke a priceless cup, but none of them broke both a vase and a cup. How many bulls broke
(a) a vase or a plate or both?
(b) a plate and a cup?
(c) a vase only?

6.3. The multiplication Principle.

Proposition. If a task consists of a sequence of k choices in which there are n_1 selections for the 1st choice, n_2 selections for the 2nd choice, $\ldots$, n_k selections for the kth choice, then the task of making these selections can be done in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ different ways.

Draw a tree diagram to illustrate this principle.

Example. How many phone numbers can be put on 555-xxxx exchange?
Example. How many four-digit PINs are possible if zero cannot be used as the first digit and no digit may be repeated? 4536
Example. Three different novels and two different manuals are to be arranged on a shelf. In how many ways can it be done if a novel is to occupy the middle position? $4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 72$
Example. Suppose that a license plate is made up of 2 letters followed by 5 digits. How many different such plates can be made?
Example. How many 7-digit phone numbers exit that contain at least 1 zero? $10^7 - 9^7 = 5,217,031$
Example. Two-digit numbers are to be made up from the digits 1, 4, 5, 7, and 9.
(a) How many such numbers exist if repetitions are allowed?
(b) if repetitions are not allowed?
(c) How many even numbers are possible?

Example. A nursery rhyme starts as follows:

As I was going to St. Ives
I met a man with seven wives.
Each wife had seven sacks.
Each sack had seven cats.
Each cat had seven kittens.

How many kittens did the traveller meet?

Example. 100,000 people are to be given an ID in which one letter is followed be n digits. What is the smallest possible value of n?

6.4. Permutations.

Definition. A permutation is an ordered arrangement of r objects chosen from n objects.
Notation. $P(n, r)$.
Definition. A factorial $n!$ is defined as $n! = n(n-1)(n-2) \cdot \cdots \cdot 1$.
Example. $0! = 1,\ 1! = 1,\ 2! = 2$, etc.
Proposition. $P(n, r) = \frac{n!}{(n-r)!}$.
Proof: There are n balls and r boxes. There are n choices for the 1st box, $n-1$ choices for the 2nd, etc. □

Example. In how many ways can 4 of 8 books be arranged on a shelf? 1680
Example. In how many ways can 5 people sit in a row?
Example. (a) An art appreciation class is asked to rank the paintings 1 through 7. How many different rankings are possible? 5040
(b) If the students are asked to rank only the top three, how many rankings are possible? 210

Example. In how many ways can 3 of 10 people be elected for the positions of president, secretary, and treasurer? 720

6.5. Combinations.

Definition. A <u>combination</u> is an unordered arrangement of r objects chosen from n objects..

Notation. $C(n, r)$ or $\binom{n}{r}$.

Proposition. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Proof: $P(n, r)$ is the number of ordered subsets; $\binom{n}{r}$ is the number of unordered subsets, which can be ordered in $r!$ ways. Therefore, $\binom{n}{r} r! = P(n, r)$. □

Example. Suppose we have three balls: A, B, and C. The number of combinations is one, while the number of permutations is six (ABC, ACB, BAC, BCA, CAB, CBA).

Example. To illustrate that $P(3, 2) = 6$ and $\binom{3}{2} = 3$, consider elements $\{a, b, c\}$. There are six permutations $a\,b \quad b\,a \quad a\,c \quad c\,a \quad b\,c$ and $c\,b$ but only three combinations $a\,b \quad a\,c$ and $b\,c$.

Example. A committee of three is to be formed from a group of 20 people. How many different committees are possible? $\binom{20}{3} = \frac{(20)(19)(18)(17!)}{3!17!} = 1,140$

Example. Consider a group of seven people. If everyone shakes hands with everyone else, how many handshakes take place? $\binom{7}{2} = 21$

Example. Ten children divide themselves into team A and team B of five each. How many different divisions are possible? $\binom{10}{5} = 252$

Example. How many 4-card hands are possible? $\binom{52}{4} = 270,725$

Proposition. The number of permutations of n objects, of which n_1 are of the first kind, n_2 are of the second kind, $\ldots$, n_k are of the kth kind, is $\frac{n!}{n_1!\,n_2!\ldots n_k!}$ where $n_1 + n_2 + \cdots + n_k = n$.
Proof: Total number of permutations is $n!$. It has to be reduced since all n_1 objects of the first kind are indistinguishable, etc. □

The above proposition can be restated.
Proposition. The number of ways to partition n objects into k groups containing $n_1, n_2, \ldots, n_k$ objects is $\frac{n!}{n_1!\,n_2!\ldots n_k!}$ where $n_1 + n_2 + \cdots + n_k = n$.
Proof: $\binom{n}{n_1}\binom{n-n_1}{n_2}\ldots\binom{n_k}{n_k} = \frac{n!}{n_1!\,n_2!\ldots n_k!}$. □

Example. A police department consists of ten officers. Five officers patrol the streets, two work at the station, and three are on reserve.
(a) How many different divisions are possible? $\frac{10!}{5!\,2!\,3!} = 2,520$

(b) How many different divisions are possible if officer Larson patrols the streets? $\binom{9}{4}\binom{5}{2} = \frac{9!}{4!\,2!\,3!} = 1,260$

Example. A committee of four is to be selected from among eight students and a professor. In how many ways can it be done if
(a) the professor cannot be on the committee? $\binom{8}{4} = 70$
(b) the professor must be on the committee? $\binom{8}{3} = 56$
(c) there are no restrictions? $\binom{9}{4} = 126$

Example. How many distinct "words" can be formed using all letters in
(a) TENNESSEE? *Solution:* There are four groups of letters: T-1, E-4, N-2, S-2. Thus, the answer is $\frac{9!}{1!\,4!\,2!\,2!} = 3,780$
(b) STATISTICS? S-3, T-3, A-1, I-2, C-1, $\frac{10!}{3!\,3!\,1!\,2!\,1!} = 50,400$

7.1. Sample Spaces.

Definition. A random experiment is a procedure that (1) can be repeated as many times as we want and (2) has a well-defined set of possible outcomes, but outcomes are uncertain on every trial.
Example. Flipping a coin: outcomes are H or T; rolling a die: outcomes are 1, 2, 3, 4, 5, or 6.
Definition. A sample space S is a set of all possible outcomes of a random experiment.
Examples. (1) Flipping a coin once. $S = \{H, T\}$
(2) Flipping a coin twice. $S = \{HH, HT, TH, TT\}$
(3) Flipping a coin three times. $S\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
(4) Tossing a die. $S = \{1, 2, 3, 4, 5, 6\}$
(5) Tossing two dice. $S = \{(1,1), (1,2), \dots, (1,6), \dots, (6,1), (6,2), \dots, (6,6)\}$
(6) Measuring the lifetime of a computer. $S = \{x : 0 \le x < \infty\}$.

Assignment of Probabilities.
Definition. Suppose the sample space S has n outcomes given by $S = \{e_1, e_2, \dots, e_n\}$. To each outcome we assign a real number $\mathbb{P}(e)$, called the probability of the outcome e, satisfying: (1) $\mathbb{P}(e) \ge 0$ for every $e \in S$, (2) $\mathbb{P}(e_1) + \cdots + \mathbb{P}(e_n) = 1$.
Example. Flip a coin. $S = \{H, T\}$, $\mathbb{P}(H) = \mathbb{P}(T) = 1/2$. Such a coin is called fair.
Example. Flip a biased coin, e.g., $\mathbb{P}(H) = 0.8, \mathbb{P}(T) = 0.2$.
Example. Toss a fair die. $\mathbb{P}(1) = \cdots = \mathbb{P}(6) = 1/6$.
Example. Toss a loaded die, e.g., $\mathbb{P}(1) = \cdots = \mathbb{P}(5) = 0.1, \mathbb{P}(6) = 0.5$.

Constructing a Probability Model of a Random Experiment.
Definition. A probability model of an experiment consists of the sample space and the assignment of probabilities to each outcome.
Example. A fair coin is tossed. Write down the probability model.
Solution: $S = \{H, T\}; \mathbb{P}(H) = \mathbb{P}(T) = 1/2$.
Example. A fair coin is flipped two times. Write down the probability model.
Solution. $S = \{HH, HT, TH, TT\}; \mathbb{P}(HH) = \cdots = \mathbb{P}(TT) = 1/4$.
Example. A fair coin is tossed until a head or three tails appear. Write down the probability model.
Solution: $S = \{H, TH, TTH, TTT\}; \mathbb{P}(H) = 1/2, \mathbb{P}(TH) = 1/4, \mathbb{P}(TTH) = 1/8, \mathbb{P}(TTT) = 1/8$.

Probability Models Involving Equally Likely Outcomes.

Definition. When the same probability is assigned to every outcome of a random experiment, the outcomes are called equally likely outcomes.
Example. One card is drawn at random from a deck of cards.
$S = \{2\clubsuit, 2\spadesuit, 2\heartsuit, 2\diamondsuit, \dots, A\clubsuit, A\spadesuit, A\heartsuit, A\diamondsuit\}; \mathbb{P}(card) = 1/52$.
Definition. An event is a subset of S.
Notation. An event is denoted A, B, C, D, E, F, G.
Example. Two coins are flipped. Event A is to see at least one head. List the outcomes in A.
Solution: $A = \{HT, TH, HH\}$.
Example. A card is drawn. An event B is to draw an ace or the queen of spades. List all the outcomes in B.
Solution: $B = \{A\clubsuit, A\spadesuit, A\heartsuit, A\diamondsuit, Q\spadesuit\}$.

Probability of an Event E in a Sample Space with Equally Likely Outcomes.
If the sample space S has n equally likely outcomes, and the event E has m outcomes, then the probability of the event E, $\mathbb{P}(E) = m/n$ =number of outcomes in E/number of all outcomes= $c(E)/c(S)$.
Example. A box contains 3 red, 4 green and 6 blue marbles. One marble is drawn at random. What is the probability to choose a green marble?
Solution: Let G=a green marble is picked, $c(G) = 4$. Each marble is equally likely to be chosen, $c(S) = 13$. Therefore, $P(G) = 4/13$.
Example. A card is drawn from a deck of cards. Find the probability that
(a) an ace is drawn.
Solution: Each card is equally likely to be drawn, so the probability to drawn one particular card is 1/52. There are four aces. Therefore, $\mathbb{P}(\text{ace}) = 4/52 = 1/13$.
(b) a heart is drawn. $\mathbb{P}(\text{heart}) = 13/52 = 1/4$.

7.2. Probability of an Event.

Definition. An event is a subset of the sample space S. A simple event is an event consisting of exactly one outcome $E = \{e\}$.
Definition. The probability of a simple event $\mathbb{P}(E) = \mathbb{P}(e)$. The probability of an event $E = \{e_1, e_2, \dots, e_n\}$ is $\mathbb{P}(E) = \mathbb{P}(e_1) + \mathbb{P}(e_2) + \cdots + \mathbb{P}(e_n)$.
Example. A loaded coin is flipped twice. $\mathbb{P}(H) = .7$. Find the probability that the outcomes are identical.
Solution: $A = \{HH, TT\}, \mathbb{P}(A) = \mathbb{P}(HH) + \mathbb{P}(TT) = (.7)(.7) + (.3)(.3) = .58$.
Example. One person is chosen from a group of students. The probability of each person being selected is given in the following table:

	Math	Art	Biology
Freshman	.1	.08	.17
Sophomore	.22	.3	.13

Find the probability that
(1) a freshman is selected.
(2) an art major is selected.
(3) a freshman math major or a sophomore biology major is selected.

Properties of the Probability.
(1) $0 \leq \mathbb{P}(E) \leq 1$,
(2) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(S) = 1$,
(3) Additive Rule: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ for any events A and B,
Definition. A compliment of an event E, denoted $\bar{E}$, is the event consisting of all the outcomes that are not in E.
(4) $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$.
Definition. Events F and G are called mutually exclusive or disjoint if $F \cap G = \emptyset$. The probability of the union of two disjoint events is $\mathbb{P}(F \cup G) = \mathbb{P}(F) + \mathbb{P}(G)$.
Example. If $\mathbb{P}(A) = .6$, $\mathbb{P}(B) = .3$, and $\mathbb{P}(A \cap B) = .1$, find
(1) $\mathbb{P}(\bar{A})$,
(2) $\mathbb{P}(A \cup B)$,
(3) $\mathbb{P}(A \cap \bar{B})$,
(4) $\mathbb{P}(\bar{A} \cap \bar{B})$.
Example. Of the shirts sold in a clothing store, 5% have either faulty seams or a missing button, 2% have faulty seams, and .5% have both flaws. What is the probability that a shirt will have a missing button?
Example. One card is drawn. A=jack, B=heart. Are A and B disjoint? Give an example of disjoint events.

Example. A coin is tossed four times. Find the probability that heads turn up at least once.

Example. At a furniture plant, 2% of desks has structural defects, 3% have finish defects, and 1% have both. Find the probability that a randomly cho-

sen desk has
(a) at least one defect,
(b) neither kind of defect.

Odds.
If the odds for (or in favor of) E are a to b, then $\mathbb{P}(E) = \frac{a}{a+b}$.
If the odds against E are a to b, then $\mathbb{P}(E) = \frac{b}{a+b}$.
Example. If the odds for rain today are estimated to be 1 to 3, what is the probability that
(a) it rains today? $\frac{1}{1+3} = 1/4$
(b) it doesn't rain today? $\frac{3}{1+3} = 3/4$.
Example. A single card is dealt. Find the odds for the card being a king. $\mathbb{P}(\text{king}) = 1/13$, odds are 1 to 12.

7.3. Probabilities Problems Using Counting Techniques.

Example. From a group of five women and seven men, randomly choose five people. What is the probability that
(a) two women and three men are chosen?

$$\frac{\binom{5}{2}\binom{7}{3}}{\binom{12}{5}} = \frac{(10)(35)}{792} = 0.44$$

(b) women are not chosen?

$$\frac{\binom{5}{0}\binom{7}{5}}{\binom{12}{5}} = \frac{21}{792} = 0.03$$

(c) Mr. N. is chosen?

$$\frac{\binom{11}{4}}{\binom{12}{5}} = \frac{330}{792} = 0.42$$

(d) Mr. N. is the only man chosen?

$$\frac{\binom{5}{4}}{\binom{12}{5}} = \frac{5}{792} = 0.006$$

Example. From among three conservatives and five liberals, a committee of three is to be selected. What is the probability that at least two are liberals? $\mathbb{P}(\text{exactly two liberals}) + \mathbb{P}(\text{exactly three liberals}) = \frac{\binom{3}{1}\binom{5}{2}}{\binom{8}{3}} + \frac{\binom{3}{0}\binom{5}{3}}{\binom{8}{3}} = \frac{(3)(10)}{56} + \frac{10}{56} = \frac{40}{56} = 0.71$.

Example. Four cards are dealt. Find the probability that at most two are face cards.
$\mathbb{P}(\text{at most two face cards}) = 1 - \big[\mathbb{P}(\text{three face cards}) + \mathbb{P}(\text{four face cards})\big] = 1 - \big[\frac{\binom{12}{3}\binom{40}{1}}{\binom{52}{3}} + \frac{\binom{12}{4}}{\binom{52}{4}}\big] = 1 - \big[\frac{(220)(40)}{270725} + \frac{495}{270725}\big] = 1 - \frac{9295}{270725} = 0.97$.

Example. A coin is tossed four times. What is the probability that
(a) exactly two heads turn up?
$\mathbb{P}(\text{exactly two heads}) = \frac{\binom{4}{2}}{2^4} = \frac{6}{16} = 0.375.$
(b) at least one tail turns up?
$\mathbb{P}(\text{at least one tail}) = 1 - \mathbb{P}(\text{no tails}) = 1 - \frac{1}{16} = \frac{15}{16} = 0.9375.$

Example. License plates are made with three letters followed by three digits. What is the probability that a license plate
(a) starts with the letter A and ends with the digit 8?

$$\frac{(26)(26)(10)(10)}{(26)^3(10)^3} = \frac{1}{260} = 0.004$$

(b) has no letter or digit repeated?

$$\frac{(26)(25)(24)(10)(9)(8)}{(26)^3(10)^3} = 0.64$$

(c) doesn't contain the letter Z or the digit 0?

$$\frac{(25)^3 9^3}{(26)^3(10)^3} = 0.65$$

Example. (Birthday Problem). Find the probability that in a group of r people at least two have the same birthday.

$$\mathbb{P} = 1 - \frac{(365)(364)\dots(365-r+1)}{(365)^r}.$$

If, e.g., $r = 5$, $\mathbb{P} = 0.027$.

7.4. Conditional Probability.

Definition. A <u>conditional</u> probability of an event E given that an event F happened is

$$\mathbb{P}(E|\,F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Example. In a group of 200 students, 40 are taking English, 50 are taking Math, and 12 are taking both. Given that a student is taking English, what is the probability that he is taking Math?

Example. Draw a marble from a box containing 3 green, 1 white, and 5 black marbles. If the drawn marble is not white, find the probability that it is green. Do the calculations in two ways:

(a) by definition of conditional probability.

(b) by reduction of the sample space.

Example. A random sample of 200 adults are classified by gender and their level of education attained.

Education	**Male**	**Female**
Elementary	38	45
Secondary	28	50
College	22	17

If a person is picked at random from this group, find the probability that

(a) the person is a male, given that the person has a secondary education.

(b) the person does not have a college degree, given that the person is a female.

Example. $\mathbb{P}(A) = .4$, $\mathbb{P}(B) = .7$, $\mathbb{P}(A \cap B) = .3$. Find probabilities
(a) $\mathbb{P}(A|\, B)$,
(b) $\mathbb{P}(\bar{A}|\, B)$,
(c) $\mathbb{P}(A|\, \bar{B})$,
(d) $\mathbb{P}(\bar{A}|\, \bar{B})$.

Product Rule.

$$\mathbb{P}(E \cap F) = \mathbb{P}(E|\, F)\,\mathbb{P}(F) = \mathbb{P}(F|\, E)\,\mathbb{P}(E)$$

Example. Two cards are drawn sequentially without replacement. What is the probability that the first is an ace and the second is a king?
Solution: $\mathbb{P}$(ace first) $= 4/52$, $\mathbb{P}$(king second| ace first) $= 4/51$, $\mathbb{P}$(ace first and king second) $= (4/52)(4/51) = 4/663$.

Example. A contractor buys lumber from two suppliers: 60% from supplier A and 40% from supplier B. Upon delivery he discovers that 2% of the lumber from A is damaged and 4% from B is damaged. What is the probability that a randomly chosen piece of lumber came from supplier A and was damaged? Draw a tree diagram.

Example. Two marbles are drawn, in succession without replacement, from a box containing 2 red and 5 white balls. What is the probability that the first ball is white and the second one is red?

7.5. Independent Events.

Definition 1. Two events E and F are <u>independent</u> iff $\mathbb{P}(E|\, F) = \mathbb{P}(E)$.

Definition 2. Two events E and F are <u>independent</u> iff $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$.

The Proof that the Two Definitions are Equivalent.
Def.1 $\Rightarrow$ Def.2. Suppose $\mathbb{P}(E|\, F) = \mathbb{P}(E)$. Then $\mathbb{P}(E \cap F) \overset{product\ rule}{=} \mathbb{P}(E|\, F)\mathbb{P}(F) \overset{assumption}{=} \mathbb{P}(E)\mathbb{P}(F)$.
Def.2 $\Rightarrow$ Def.1. Suppose now $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$. Then $\mathbb{P}(E|\, F) \overset{def}{=} \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} \overset{assumption}{=} \frac{\mathbb{P}(E)\mathbb{P}(F)}{\mathbb{P}(F)} = \mathbb{P}(E)$. $\square$

Example. Draw a card. $A = \{\text{ace}\}$, $B = \{\text{black}\}$. Are A and B independent?
Solution 1: $\mathbb{P}(A) = 4/52 = 1/13$, $\mathbb{P}(B) = 26/52 = 1/2$. $\mathbb{P}(A \cap B) = \mathbb{P}(\text{black ace}) = 2/52 = 1/26$. Therefore, $\mathbb{P}(A \cap B) = 1/26 = \mathbb{P}(A)\mathbb{P}(B) = (1/13)(1/2) \implies A$ and B are independent.
Solution 2: $\mathbb{P}(A|\, B) = 2/26 = 1/13 = \mathbb{P}(A)$ or $\mathbb{P}(B|\, A) = 1/2 = \mathbb{P}(B) \implies$ A and B are independent.

Example. Events C and D are independent. $\mathbb{P}(C) = 1/4$, $\mathbb{P}(D) = 1/2$. Find $\mathbb{P}(C \cup D)$.
Solution: $\mathbb{P}(C \cup D) = \mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C \cap D) = \mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C)\mathbb{P}(D) = 1/4 + 1/2 - (1/4)(1/2) = 5/8$.

Definition. More than two events are <u>independent</u> if occurrence of some of them doesn't change the probabilities of the others. That is, $E_1, \dots, E_n$ are independent iff

$$\begin{cases} \mathbb{P}(E_i \cap E_j) = \mathbb{P}(E_i)\mathbb{P}(E_j), \quad i, j = 1, \dots, n, i \neq j \\ \mathbb{P}(E_i \cap E_j \cap E_k) = \mathbb{P}(E_i)\mathbb{P}(E_j)\mathbb{P}(E_k), \quad i, j, k = 1, \dots, n, i \neq j \neq k \\ \dots \\ \mathbb{P}(E_1 \cap \dots \cap E_n) = \mathbb{P}(E_1) \dots \mathbb{P}(E_n) \end{cases}$$

Proposition. If $E_1, E_2, \dots, E_n$ are independent, then $\mathbb{P}(E_1 \cap E_2 \cap \dots \cap E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2) \dots \mathbb{P}(E_n)$.

Example. A string of lights wired in series contains 5 lights. Lights fail independently with probability 0.02. What is the probability that the string of lights is bright?
Solution: $\mathbb{P}(\text{all 5 lights work}) = \mathbb{P}(\{1st\ works\} \cap \{2nd\ works\} \cap \dots \cap \{5th\ works\}) = (0.98)(0.98) \dots (0.98) = 0.9$.

Example. Can mutually exclusive events be independent?

Example. A general can plan a campaign to fight one major battle or three small battles. The probabilities of winning the battles are 0.6 and 0.8, respectively. The small battles are won independently. Which strategy should he choose?

Solution: $\mathbb{P}(\text{win 3 small battles}) = (0.8)^3 = 0.512 < 0.6 = \mathbb{P}(\text{win major battle}) \implies$ fight major battle.

Example. Here is the composition of Congress by party and seniority.
(a) Are the party and seniority independent?

Seniority	Democrat	Republican
<2years	0.05	0.05
2-9 years	0.35	0.25
$\geq$ 10 years	0.2	0.1

(b) What would the probabilities in the body of the table be if party and seniority were independent?

Seniority	Democrat	Republican	Total
<2years	**0.06**	**0.04**	0.1
2-9 years	**0.36**	**0.24**	0.6
$\geq$ 10 years	**0.18**	**0.12**	0.3
Total	0.6	0.4	

Example. A town has two fire engines operating independently. The probability that an engine is available is 0.96. What is the probability that
(a) both engines are available?
Solution: $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = (0.96)^2 = 0.92.$
(b) neither is available?
Solution: $\mathbb{P}(\bar{A}_1 \cap \bar{A}_2) = \mathbb{P}(\bar{A}_1)\mathbb{P}(\bar{A}_2) = (0.04)^2 = 0.0016.$

8.1. Bayes' Formula.

Definition. Events $A_1,\ A_2, \ldots, A_n$ <u>partition</u> the sample space S if
(i) A_i's are not empty: $A_i \neq \emptyset$ for any $i = 1, \ldots, n$,
(ii) A_i's are disjoint: $A_i \cap A_j \neq \emptyset$ for any $i, j,\ \ i \neq j$,
(iii) $A_1 \cup A_2 \cup \cdots \cup A_n = S$.

"Unconditional" Probability.
Let E be an event in S and $A_1,\ A_2, \ldots, A_n$ be a partition of S. Then $E = (E \cap A_1) \cup (E \cap A_2) \cup \cdots \cup (E \cap A_n)$. Since $E \cap A_i$ are disjoint, $\mathbb{P}(E) = \mathbb{P}(E \cap A_1) + \cdots + \mathbb{P}(E \cap A_n) \overset{\textit{product rule}}{=} \mathbb{P}(E|\, A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(E|\, A_n)\mathbb{P}(A_n)$.

The Bayes Rule.
For any partition $A_1, \ldots, A_n$, any event E, and any $i = 1, \ldots, n$,

$$\mathbb{P}(A_i|\, E) = \frac{\mathbb{P}(E|\, A_i)\mathbb{P}(A_i)}{\mathbb{P}(E|\, A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(E|\, A_n)\mathbb{P}(A_n)}.$$

Proof:

$$\mathbb{P}(A_i|\, E) = \frac{\mathbb{P}(E \cap A_i)}{\mathbb{P}(E)} = \{product\,rule,\ \text{“}unconditional\text{''}\ probability\}$$

$$= \frac{\mathbb{P}(E|\, A_i)\mathbb{P}(A_i)}{\mathbb{P}(E|\, A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(E|\, A_n)\mathbb{P}(A_n)}. \quad \square$$

Definition. The probability $\mathbb{P}(A_i)$ is called a priori probability of event A_i. The probability $\mathbb{P}(A_i|\, E)$ is called a posteriori probability.

Example. Two balls are randomly drawn, in succession and without replacement, from a box containing two red and five white balls. What is the probability that
(a) the second ball drawn is red?
Solution: Draw the tree diagram. $(2/7)(1/6) + (5/7)(2/6) = 2/7$.
(b) the first ball is is white, given that the second ball is red?
Solution: $\mathbb{P}(\text{1st white}|\, \text{2nd red}) = \frac{(5/7)(2/6)}{(2/7)(1/6)+(5/7)(2/6)} = 5/6$.

Historical Note.
The Reverend Thomas Bayes (1702-1761) was an English clergyman who discovered the formula for what he called “reverse” probabilities in a sample space. His mathematical work was a hobby, and his famous formula was not published until after his death. A correct proof was not given until the 1930s.

Example. A department store made a study of the personal checks it received for payment of goods. It discovered that 40% of the checks with insufficient funds had the wrong date on them, while only 2% of all good checks had the wrong date on them. It also found that 0.5% of all checks received had insufficient funds to cover them. If a clerk in this store receives a personal check form a customer, what is the probability that the check has insufficient funds, given that it has the wrong date?
Solution: $\mathbb{P}(\text{insufficient funds}|\, \text{wrong date}) = \frac{(0.005)(0.4)}{(0.005)(0.4)+(0.995)(0.02)} = 0.09$.

Example. An insurance company believes that people can be divided into two classes – accident prone and non-accident prone. An accident-prone person will have an accident within a year with probability 0.4, a non-accident-prone person – with probability 0.2. If 30% of the population is accident-prone,
(a) what is the probability that a new policyholder will have an accident within a year?
Solution: $\mathbb{P}(\text{accident}) = (0.3)(0.4) + (0.7)(0.2) = 0.26$.
(b) Suppose that the new policyholder has an accident. What is the probability that he is accident-prone?
Solution: $\mathbb{P}(\text{accident-prone}|\ \text{accident}) = \frac{(0.4)(0.3)}{0.26} = 0.46$.

Example. A large industrial firm uses 3 local motels to provide overnight accommodations for its clients. From past experience it is known that 20% of the clients are assigned rooms at the Ramada Inn, 50% at the Sheraton,

and 30% at the Lakeview Motor Lodge. If the plumbing is faulty in 5% of the rooms at the Ramada Inn, in 4% of the rooms at the Sheraton, and in 8% of the rooms at the Lakeview Motor Lodge, what is the probability that a person with a room having faulty plumbing was assigned accommodations at the Lakeview Motor Lodge?
Solution: $\mathbb{P}(\text{Lakeview}|\,\text{faulty plumbing}) = \frac{(0.08)(0.3)}{(0.05)(0.2)+(0.04)(0.5)+(0.08)(0.3)} = 0.44.$

8.2. The Binomial Probability Model.

Definition. A random experiment is called a Bernoulli trial if
(i) there are only two possible outcomes, called "success" and "failure."
(ii) the probability of a success is constant.

Definition. The Binomial Probability Model is a sequence of a fixed number n of independent Bernoulli trials, each with probability of a success p. Thus, the Binomial model has parameters n and p.
Proposition. The probability of exactly k successes in n trials with probability of a success p is

$$b(n,k;p) = \binom{n}{k} p^k\,(1-p)^{n-k}.$$

Proof: n trials result in k successes and $n-k$ failures. The probability of such an outcome is $p^k\,(1-p)^{n-k}$. There are $\binom{n}{k}$ outcomes like that. □

Example. A coin is tossed 5 times. What is the probability that
(a) exactly two heads appear? $\binom{5}{2}(1/2)^5 = 5/16$.
(b) at most two heads appear? $(1/2)^5(1+5+10) = 1/2$.

Example. A multiple-choice quiz of four questions is given. Each question has five possible answers. If a student guesses at all the answers, find the probability that three out of four are correct. $\binom{4}{3}(1/5)^3(4/5) = 16/625$.

Example. The probability that a marksman will hit a target is 0.7. Find the probability that he will hit the target less than 4 times out of five attempts. $1-\binom{5}{4}(0.7)^4(0.3)-(0.7)^5 = 1-0.36015-0.16807 = 0.47$.

Example. If a family has four children, is it more likely they will have two boys and two girls or three of one sex and one of the other? $\binom{4}{2}(1/2)^4,\ 2\binom{4}{3}(1/2)^4$.

Example. A tour company uses a bus with capacity of ten passengers but sells twelve tickets. One person out of six is a no-show. What is the probability that everyone who shows up for the tour will be accommodated? $1-\binom{12}{11}(5/6)^{11}(1/6)-(5/6)^{12} = 1-0.269-0.112 = 0.619$.

Example. The gunner on a small assault boat fires six missiles at an attacking plane. Each has a 20% chance of being on target. If two or more of

the shells find their mark, the plane will crash. At the same time, the pilot of the plane fires 10 air-to-surface rockets, each of which has a 0.05 chance of critically disabling the boat. Would you rather be on the plane or the boat?
$1-\binom{6}{0}(0.8)^6-\binom{6}{1}(0.2)(0.8)^5=1-0.26-0.39=0.35,\ 1-(0.95)^{10}=0.40.$

8.5. Random Variables.

Definition. A random variable is a rule that assigns a number to every outcome in the sample space.

Notation. X, Y, Z, W, U, V.

Example. Two coins are tossed. Let X be the number of heads that appear. Is X a random variable?
Solution: X takes on values 0, 1, or 2 with probabilities $\mathbb{P}(X=0)=\mathbb{P}(\{TT\})=1/4$, $\mathbb{P}(X=1)=\mathbb{P}(\{HT, TH\})=1/2$, $\mathbb{P}(X=2)=\mathbb{P}(\{HH\})=1/4$.

Definition. The list of values of a random variable with the respective probabilities is called the probability distribution of X.
Notation. The values are denoted $x_1, x_2, \ldots, x_n$, and the probabilities are given by $p\,(x)=\mathbb{P}(X=x)$.

Properties of the Probability Distribution.

$$(i)\ \ 0\le p\,(x)\le 1 \text{ for } x=x_1,\ x_2,\ldots,x_n$$
$$(ii)\ \ p\,(x_1)+p\,(x_2)+\cdots+p\,(x_n)=1.$$

Example *. Two people are randomly selected from a group of five men and four women. A r.v. Y is the number of women selected. Find the probability distribution of Y.
Solution: $p\,(0)=10/36,\ p\,(1)=20/36,\ p\,(2)=6/36.$

Example **. Let N be the number of male children in a family of 3. Find the probability distribution of N.
Solution: $p\,(0)=1/8,\ p\,(1)=3/8,\ p\,(2)=3/8,\ p\,(3)=1/8.$

Definition. The expected value of X is $\mathbb{E}(X)=x_1\,p\,(x_1)+\cdots+x_n\,p\,(x_n)$.

Example. Find the expected value of Y in Example *.
Solution: $\mathbb{E}(Y)=8/9$.

Example. Find $\mathbb{E}(N)$ in Example **.
Solution: $\mathbb{E}(N)=3/2$.

Example. The number of defects in a machine-made product is a random variable X such that

x	0	1	2	3	4
p (x)	.1	.2	.3	.3	?

(a) Find $p(4) = .1$.
(b) Find the expected number of defects. 2.1.

Example. In an auto repair shop 15% of the cars need minor repairs averaging \$20, 65% need moderate repairs averaging \$130, and 20% need major repairs averaging \$700. What is the expected cost of repair of a car selected at random?

Example. A couple plans to have children until one boy or four girls are born. What is the expected number of children they have?

8.3. Examples of Expected Value.

Example. I offer you a once-in-a-lifetime opportunity to play the following game. We flip a coin three times. If we see exactly two heads, I'll pay you \$10. Otherwise, you pay me \$7. Will you be willing to play?
Solution: Let G be your gain. Then $\mathbb{P}(G = 10) = \mathbb{P}(\{HHT, HTH, THH\}) = 3/8$, $\mathbb{P}(G = -7) = 5/8$, $\mathbb{E}(G) = 30/8 - 35/8 = -5/8$. Don't play. Play if you pay me \$6. Then $\mathbb{E}(G) = 0$, a <u>fair</u> game.

Example. A psychic runs the following ad in a magazine: Expecting a baby? Renowned psychic will tell you the sex of the unborn child from any photograph of the mother. Cost \$10. Moneyback guarantee. This may be a profitable con game. Suppose that the psychic simply replies"girl" to each inquiry. In the worst case, everyone who has a boy will ask for her money back. Find the expected value of the psychic's profit.
Solution: Let X be the psychic's profit. Then $\mathbb{P}(X = 10) = \mathbb{P}(girl) = 1/2$, $\mathbb{P}(X = 0) = \mathbb{P}(boy) = 1/2$, $\mathbb{E}(X) = \$5$.

Example. The Connecticut State Lottery awards at random, for each 100,000 one-dollar tickets sold,

$$\begin{array}{rrl} 1 & \$5,000 & \text{prize,} \\ 18 & \$200 & \text{prizes,} \\ 120 & \$25 & \text{prizes,} \\ 270 & \$20 & \text{prizes.} \end{array}$$

What is the expected value of the winnings of one ticket in this lottery? Do you want to play?
Solution: Let W be the winning of one ticket. Then $\mathbb{P}(W = 5000) = 1/10^5$, $\mathbb{P}(W = 200) = 18/10^5$, $\mathbb{P}(W = 25) = 120/10^5$, $\mathbb{P}(W = 20) =$

$270/10^5$, $\mathbb{E}(W) = 0.17$.

Example. A grab-bag contains 6 packages worth \$2 each, 11 packages worth \$3, and 8 packages worth \$4 each. Is it reasonable to pay \$3.50 for the option of selecting one of these packages at random?
Solution: Let Z be the price of a drawn package. Then $\mathbb{P}(Z = 2) = 6/25$, $\mathbb{P}(Z = 3) = 11/25$, $\mathbb{P}(Z = 4) = 8/25$, $\mathbb{E}(Z) = \$3.08$. It is worth to pay \$3.08 to play this game but not \$3.50.

Expected Value for Bernoulli Trials.
Definition. Let X be the number of successes in the Binomial probability model. Then, X is said to have a <u>Binomial</u> distribution. $\mathbb{P}(X = k) = b(n, k; p)$.

Proposition. If X has a binomial distribution, then $\mathbb{E}(X) = np$.

Example. A quiz has five multiple-choice questions with four possible answers to each. If a student guesses the answers, how many questions will he get correctly on average?
Solution: Let X be the number of correct guesses. Then $X \sim B(5, 1/4)$, $\mathbb{E}(X) = 5/4$.

Example. Test show that about 4% of the people who take a particular drug are subject to side effects. Of 20 people taking the drug, how many do we expect will have a side effect?

9.1. Data and Sampling.

Definition. A <u>variable</u> is a measurable characteristic of a person or a thing.
Example. Height, weight, gender, income.
Definition. <u>Data</u> is a set of observations of a particular variable.
Definition. <u>Statistics</u> provides methods to collect data, organize them in a meaningful way, and interpret and report conclusions.
Definition. <u>Quantitative</u> (or numerical) variable is a variable for which arithmetic operations make sense.
Definition. <u>Categorical</u> (or qualitative) variable records into which of several categories a person or thing falls.
Definition. <u>Discrete</u> variable is a variable that assumes a finite or countably infinite set of values.
Definition. <u>Continuous</u> variable assumes values in an interval.
Example. Is the variable discrete or continuous?
(a) the number of car accidents per year in CA.

(b) the length of time to play 18 holes of golf.

(c) the amount of milk produced yearly by a particular cow.

(d) the number of eggs laid each month by a hen.

(e) the number of building permits issued each month in a city.

(f) the weight of grain produced per acre.
Definition. A population is the entire group for which we would like to obtain data.
Definition. A sample is a subcollection of a population.
Definition. A simple random sample is a sample for which every member of a population is equally likely to be chosen. Otherwise, it is called a biased sample.
Example. A newspaper article about an opinion poll says that 43% of Americans approve of the president's overall job performance. The poll was based on phone interviews with 1210 adults from around the U.S.
(a) What variable did this poll measure?
(b) What is the population?
(c) What is the sample?
(d) Is it a simple random sample?
(e) How else can the poll be conducted?

9.2. Bar Graph. Pie chart.

Categorical data can be presented graphically by a bar graph or a pie chart.
Example. A survey of 142 students on their usual method of transportation to class gave the following data.

Method	Frequency	Percentage
Car	22	16%
Bike	43	30%
Walk	61	43%
Scooter	11	8%
Blades	5	4%

Example.

Items in Family Budget	Percentage of Income
Food	25%
Housing	35%
Utilities	22%
Clothing	12%
Recreation	6%

9.3. Organizing Data.
Quantitative data can be displayed by drawing a histogram.

Example. Weekly salaries of 20 people are recorded:

$300	500	475	325	225
175	275	375	425	425
305	180	400	525	385
500	292	305	390	475

Step 1. Compute the range of the data.
Definition. range=max-min.
In this example, range=525-175=350.

Step 2. Divide the range into intervals of equal length, called class intervals.
In our example, $350=(7)($50).
Then compute the frequency of the data for each class interval. Construct a frequency table.

Class Interval	Frequency, f
[175, 225)	2
[225, 275)	1
[275, 325)	5
[325, 375)	1
[375, 425)	4
[425, 475)	2
[475, 525)	4
[525, 575)	1

Definition. The lower class limit is the left-end point of the class interval.
Definition. The upper class limit is the right-end point of the class interval.
Definition. The midpoint m=(upper class limit + lower class limit)/2.
Definition. The class width=upper class limit - lower class limit.
In our example, $[175, 225)$ lower class limit=175, upper class limit = 225, m=200, class width = 50.
Definition. Data presented by a frequency table is called grouped data.

Step 3. Construct the histogram.

Definition. A frequency polygon is a line graph connecting the midpoints of the tops of the histogram bars.
Definition. A cumulative frequency of a class interval is the sum of the frequencies of all intervals less than or equal to this interval. Cumulative frequencies are presented in a cumulative frequency table.

In our example,

Class Interval	Frequency, f	Cumulative Frequency, cf
[175, 225)	2	2
[225, 275)	1	3
[275, 325)	5	8
[325, 375)	1	9
[375, 425)	4	13
[425, 475)	2	15
[475, 525)	4	19
[525, 575)	1	20

Definition. A cumulative frequency distribution is a line graph of cumulative frequency plotted against the upper class limit of each interval.

9.4. Measures of Central Tendency.

Definition. The mean (or sample mean) of n observations $x_1, x_2, \ldots, x_n$ is $\bar{x} = (x_1 + \cdots + x_n)/n$.
Example. A set of observations is 3, 2, 4, 0, 1. $\bar{x} = 10/5 = 2$.

Definition. The mean for grouped data is $\bar{x} = (f_1\, m_1 + \cdots + f_k\, m_k)/n$ where k is the number of classes and $n = f_1 + \cdots + f_k$ is the total number of observations.
In our example, $k = 8,\ n = 20,\ m_1 = 200, m_2 = 250, \ldots, m_8 = 550,\ f_1 = 2,\ f_2 = 1, \ldots,\ f_8 = 1,\ \bar{x} = 7550/20 = 377.5$.

Definition. The median of an ordered set of n observations is the middle value if n is odd, and is the average of the middle two observations if n is even.
Example. 0,5,0,9,6,2, median=3.5; 2,4,8,6,4,12,2, median=4.

Definition. The mode of a set of observations is the most frequent observation.
Example. 0,5,0,9,6,2, mode=0; 5,7,3,4,6,8,9, no mode; 2,4,8,6,4,12,2, modes=2 and 4 – bimodal.

9.5. Measures of Dispersion.

Definition. The sample variance of a set of n observations is $S^2 = [(x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2]/(n-1)$.
Computational formula. $S^2 = \frac{1}{n-1}(x_1^2 + \cdots + x_n^2 - n\,\bar{x}^2)$.
Definition. The sample standard deviation $S = \sqrt{\text{Variance}} = \sqrt{S^2}$.
Example. 5,8,1,4,0, $S^2 = 10.3$, $S = 3.21$.
Definition. The variance for grouped data is $S^2 = \frac{1}{n-1}[(m_1 - \bar{x})^2\, f_1 + \cdots + (m_k - \bar{x})^2\, f_k]$.
In our example, $S^2 = 11177.63,\ S = 105.72$.

9.6. The Normal Distribution.

Definition. A Normal distribution $N(\mu, \sigma)$ is given by a bell-shaped curve which is symmetric around μ. If a random variable X has a Normal distribution with mean μ and standard deviation σ, then $\mathbb{P}(a \leq X \leq b)$ = area under the curved above the interval (a, b).
Definition. A standard normal curve is $N(0, 1)$ curve. A standard normal random variable Z has $N(0, 1)$ distribution.
Properties of a Normal(μ, σ) curve:
(i) symmetric around mean μ,
(ii) the total area under the curve is one,
(iii) the change-of-curvature points are $\mu \pm \sigma$.

Example. $\mathbb{P}(Z < 0) = \mathbb{P}(Z > 0) = 0.5$,
$\mathbb{P}(0 \leq Z < 0.13) = 0.0517$,
$\mathbb{P}(Z < 0.13) = 0.5 + 0.0517 = 0.5517$,
$\mathbb{P}(Z > 1.27) = 0.5 - 0.398 = 0.102$,
$\mathbb{P}(Z < -1.27) = \mathbb{P}(Z > 1.27) = 0.102$,
$\mathbb{P}(-1.27 < Z < 0) = \mathbb{P}(0 < Z < 1.27) = 0.398$,
$\mathbb{P}(-2 < Z < 2) = (2)(0.4772) = 0.9544$,
$\mathbb{P}(2 < Z < 3) = \mathbb{P}(0 < Z < 3) - \mathbb{P}(0 < Z < 2) = 0.4987 - 0.4772 = 0.0215$.

Definition. If x is an observed value of a $N(\mu, \sigma)$ random variable, then the Z-score is defined by $Z = \frac{x-\mu}{\sigma}$.
Definition. If X is $N(\mu, \sigma)$, then $Z = \frac{X-\mu}{\sigma}$ has $N(0, 1)$ distribution.

Example. X is $N(3, 0.7)$.
$\mathbb{P}(X > 3.2) = \mathbb{P}(\frac{X-\mu}{\sigma} > \frac{3.2-3}{0.7}) = \mathbb{P}(Z > 0.29) = 0.3859$,
$\mathbb{P}(2.3 < X < 3.7) = \mathbb{P}(-1 < Z < 1) = 0.6826$.

Example. A student took an SAT test, for which the scores are N(490,75), and got 523. Another student took an ACT test, which scores are N(18,6), and got 22. Whose score is better?
Solution: It is enough to compare the standardized Z-scores. The person with the higher score performed better. The SAT Z-score is $\frac{523-490}{75} = 0.44$, while the ACT Z-score is $\frac{22-18}{6} = 2/3 = 0.67$. So, the ACT score is better.

Example. IQ's Y of 600 applicants of a college are normally distributed with mean 115 and standard deviation 13. If the college required an IQ of at least 95, how many of these applicants will be rejected on this basis?
Solution: $\mathbb{P}(Y < 95) = \mathbb{P}(Z < \frac{95-115}{13}) = \mathbb{P}(Z < -1.54) = \mathbb{P}(Z > 1.54) = 0.5 - 0.4382 = 0.0618$. Will be rejected about $(0.0618)(600) = 37.08 \approx 37$ applicants.

Example. The grade for an exam X is a normal random variable with mean 74 and standard deviation 7.
(a) If 12% of the class are given A's, what is the lowest possible A and the highest possible B?

Solution: Let x be such that all values above it will get an A. Then, $\mathbb{P}(X > x) = \mathbb{P}(Z > \frac{x-74}{7}) = 0.12$. Thus, $\frac{x-74}{7} = 1.175 \implies x = 82.225$. The lowest A is 83, the highest B is 82.

(b) Between which values do the middle 50% of the grades lie?

Solution: $\mathbb{P}(74 - a < X < 74 + a) = 0.5 \implies \mathbb{P}(-a/7 < Z < a/7) = 2\,\mathbb{P}(0 < Z < a/7) = 0.5 \implies 0.25 = \mathbb{P}(0 < Z < a/7) \implies a/7 \approx 0.675 \implies a \approx 4.725$. Middle 50% of the grades are between 74 ± 4.725.