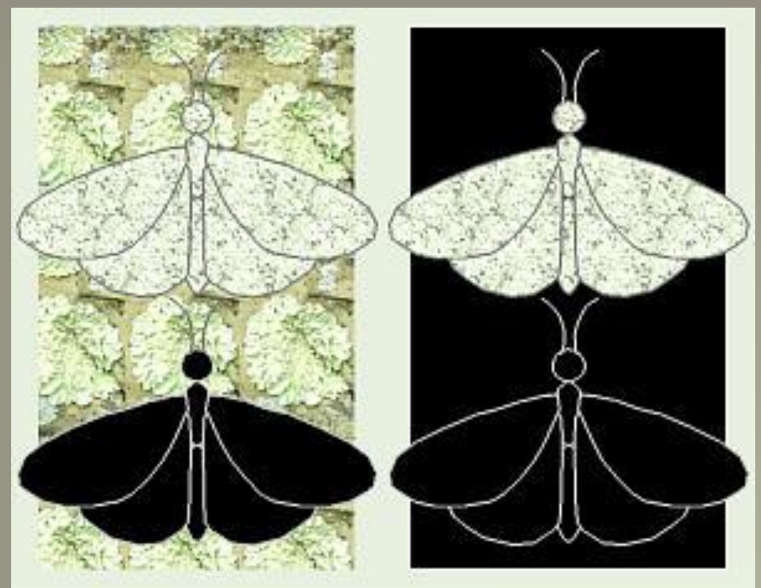

Linear Algebra & Genetics

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Introduction

Throughout the study of Genetics there are many factors that take part in why an organism looks and acts the way that they do, both in their physical shape and also in their genes. In the study of Genetics an organisms genes change to fit their surroundings either through "natural selection" or to adapt to their environment. Both are major causes for these adaptations in which the dominant traits are inherited and recessive traits are "lost" throughout their existence.

One such example is the famous Peppered Moth. There were two variations of the Peppered Moth one was lightly colored and the other was darker. Due to the Industrial Revolution the environment became full of soot from the nearby factories and the moths were effected by this. Darker moths were harder to see by predators and were picked off at a much smaller rate than the lighter colored moths because they were much more visible on the darkened trees. In this demonstration the dominant gene would be to make the moth dark, and the recessive gene would be to make the moth lighter. We can use Matrices to examine both the probability of each type of moth to survive, and predict what will happen in the future as long as the conditions remain the same.

Key Terms

Alleles – variations of a gene

Dominant/Recessive – alleles can either be dominant or recessive. A dominant allele is represented by an upper-case letter and a recessive allele is represented by a lower-case letter. As the name implies, a dominant allele will trump over its recessive counterpart. This play an important role in an organisms' phenotype.

Phenotype – refers to the observable physical characteristics of an organism. (e.g. brown or blue eyes, tall or short)

Genotype - genetic makeup of an organism. (e.g. AA, Aa, aa)

Method

Let A be the dominant allele and a be the recessive allele.

A Moth with genotype AA will have a dark color phenotype .
A Moth with genotype Aa will have a dark color phenotype.
A Moth with genotype aa will have a light color phenotype.

First, assume the crossing of 2 moths with AA genotype. These types will always be dark and survive. Therefore their probability of an offspring to be AA, Aa, and aa, respectively, are 1, 0, 0.

Second, consider the crossing of AA and Aa. The probability of their offspring to be AA, Aa, and aa, respectively, are ½, ½, 0.

Third consider crossing AA with aa, this will result in genotype Aa. Therefore the probablity of genotypes AA, Aa, and aa, respectively, are 0, 1, 0.

Matrix A is formed by using the probability of the resulting genotype.

$$A = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

X_0 = the Initial Distribution of moths

$$x_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

One year later the distribution is…

$$x_1 = Ax_0 = \begin{bmatrix} \frac{1}{3}(1+\frac{1}{2}) \\ \frac{1}{3}(\frac{1}{2}+1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

After another year the distribution is…

$$x_2 = Ax_1 = A(Ax_0) = A^2x_0 = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}.$$

For any positive integer n, n years later

$$x_n = A^n x_0 = \begin{bmatrix} 1 & \sum_{k=1}^{n} \frac{1}{2^k} & \sum_{k=1}^{n-1} \frac{1}{2^k} \\ 0 & \frac{1}{2^n} & \frac{1}{2^{n-1}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-\frac{1}{2^{n+1}} & 1-\frac{1}{2^n} \\ 0 & \frac{1}{2^n} & \frac{1}{2^{n-1}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

As n gets larger and larger, the matrix A^n approaches to…

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $x_n = A^n x_0$ will approach to….

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Suppose you are interested in the number of moths with genotype AA, Aa, aa after 20 years? This is a lengthy calculation which may result in calculation error during the many matrix multiplications. Another approach using Diagonalization reduces computation and can be used to determine the information after x amount of years.

If the matrix A can be written as a product of an invertible matrix P and a diagonal matrix D and an inverse of P, $A = PDP^{-1}$, then the computation is much simpler. This is due to the following…

$$A^n = PD^nP^{-1} \qquad \text{for} \quad n = 1, 2, \ldots$$

and

$$D^n = \begin{bmatrix} \lambda_1 & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & \lambda_k \end{bmatrix}^n = \begin{bmatrix} \lambda_1^n & 0 & 0 & . & . & . & 0 \\ 0 & \lambda_2^n & 0 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & \lambda_k^n \end{bmatrix}$$

For matrix A the eigenvalues are $\lambda_1 = 1, \lambda_2 = ½, \lambda_3 = 0$
The corresponding eigenvectors are:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{v_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$P = [v_1|v_2|v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$x_n = PD^nP^{-1}x_0 \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

As n approaches infinity…

Acknowledgements and References:

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