Fast magnetic field penetration in a neutralized ion beam of variable Density

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Experiments at the University of California, Irvine indicate that a neutralized ion beam (NIB) propagates across a magnetic field in a time scale of less than 1 μs without a significant deflection. It is shown for a constant beam density, the nonlinear and the linear diffusion times are too fast and too slow respectively to account for the observed diffusion time. If there is a component of magnetic field normal to the surface of the beam, the Hall term becomes important and the magnetic field penetrates as dispersive whistler waves. For a constant beam density, the penetration time scales with the whistler time rather than the classical diffusion time. A one-dimensional computational calculation, using a Gaussian density profile, shows that as the magnetic field penetrates from a low to a high density region, the magnetic field amplitude grows at a slower rate in comparison with a uniform plasma peak density.

I. Introduction

Experiments involving propagation of neutralized ion beams [1] across a magnetic field indicate a magnetic field penetration time determined by the plasma Hall resistivity.

Theoretical [2-4] investigations indicate that in the presence of the perpendicular field, $\mathbf{B}_{\perp}$, the Hall resistivity enables the magnetic field to penetrate into a uniform density plasma as a whistler wave. The fast penetration of the magnetic field into a non-uniform plasma has been recently investigated by Fruchtman, Gomberoff and Armale [5-7]. Armale showed, the inclusion of a small density gradient does not significantly effect the damping. However, the magnetic field penetrates the region of lower density at a faster rate.

In this paper, we consider the behavior of whistler waves in a one-dimensional plasma slab with a Gaussian density profile. On a short time scale, the ions are assumed to be immobile.

A time dependent external magnetic field is turned on in the vacuum adjacent to the plasma slab at $t > 0$ by an external current. We further assume the possible existence of a normal field component at the slab interface. We show that the magnetic field evolution can be described by a diffusion equation with a complex diffusion coefficient.

For a Gaussian density profile, the numerical solution can be separated into real and imaginary parts. These solutions show that the magnetic field penetrates the plasma as a whistler wave, with amplitude that decreases in the direction of penetration more rapidly for the Gaussian density profile than for the constant density.

II. Basic equations

Ohm's law (neglecting the pressure gradient term) for fluid plasma is [4]:

$$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = \frac{m}{ne^2}\frac{\partial \mathbf{j}}{\partial t} + \eta \mathbf{j} + \frac{1}{nec}(\mathbf{j} \times \mathbf{B}), \tag{1}$$

where

$$\eta = \frac{1}{\sigma} = \frac{m}{ne^2\tau_{ei}}, \text{ and } \mathbf{j} \approx n_0 e(\mathbf{V} - \mathbf{v}_e),$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, η is the resistivity, σ is the conductivity, m is the electron mass, e is the electron charge, c is the velocity of light, τ_{ei} is the electron-ion collision time, $\mathbf{j}$ is the current density, $\mathbf{V}$ ($\mathbf{v}_e$) is the fluid (electron) flow velocity, and n is the plasma density which varies in the direction of the field penetration.

We consider a one dimensional slab model of a neutralized ion beam propagating with a constant ion flow velocity $\mathbf{V} \approx \mathbf{v}_i$ as shown in Figure 1. To investigate the Hall effect, we assume the existence of a constant normal field component. The motivation for this is explained in references [2,7]. The total field in its component form is:

$$\mathbf{B}(x,t) = (B_\perp, B_y(x,t), B_z(x,t)), \tag{2}$$

where $B_\perp$ is a constant component normal to the plasma slab surface in the direction of the magnetic field penetration into the plasma. Using the Maxwell's equations without the displacement current:

$$\nabla \times (B_\perp, B_y, B_z) = \frac{4\pi}{c}(0, j_y, j_z), \text{ and}$$

$$\nabla \times (E_x, E_y, E_z) = -\frac{1}{c}\frac{\partial}{\partial t}(B_\perp, B_y, B_z), \tag{3}$$

equation (1), for the y-component:

$$\frac{\partial B_y}{\partial t} = D\frac{\partial^2 B_y}{\partial x^2} + \frac{c^2 m}{4\pi e^2}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{1}{n(x)}\frac{\partial B_y}{\partial x}\right)\right] + \frac{cB_\perp}{4\pi e}\frac{\partial}{\partial x}\left(\frac{1}{n(x)}\frac{\partial B_z}{\partial x}\right), \tag{4}$$

and for the z-component:

$$\frac{\partial B_z}{\partial t} = D\frac{\partial^2 B_z}{\partial x^2} + \frac{c^2 m}{4\pi e^2}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{1}{n(x)}\frac{\partial B_z}{\partial x}\right)\right] - \frac{cB_\perp}{4\pi e}\frac{\partial}{\partial x}\left(\frac{1}{n(x)}\frac{\partial B_y}{\partial x}\right), \tag{5}$$

where

$$D = \frac{c^2\eta}{4\pi}$$

is the standard diffusion coefficient, which is essentially constant because the resistivity has a weak logarithmic dependence on density n. The second term on the right-hand side of the above equations is due to the electron inertia. Neglecting the inertia term [7] and introducing a complex quantity $\tilde{B} = B_y + iB_z$, equations (4) and (5) are coupled to obtain:

$$\frac{\partial \tilde{B}}{\partial t} = D\frac{\partial^2 \tilde{B}}{\partial x^2} - \frac{icB_\perp}{4\pi e}\frac{\partial}{\partial x}\left(\frac{1}{n(x)}\frac{\partial \tilde{B}}{\partial x}\right). \tag{6}$$

We apply a time dependent external magnetic field as a boundary condition, at the vacuum-plasma interface $x = \pm L/2$. This equation is to be solved in the domain $-L/2 \leq x \leq L/2$. The density will be defined only in this domain as constant or a Gaussian.

The boundary condition is:

$$\tilde{B}_s\left(\pm \frac{L}{2}, t\right) = \tilde{B}_s(t) = \tilde{B}_0\left(1 - e^{-t/\tau_R}\right) \text{ for } t > 0,$$

where τ_R is the rise time of the applied field. The time variation of $\tilde{B}_s$ induces plasma currents in the slab which produce an internal field defined such that $\tilde{B}(x,t) = \tilde{B}_s(t) + \tilde{b}(x,t)$, with the initial condition $\tilde{B}_s(0) = \tilde{b}(x,0) = 0$. We assume a density profile such that:

$$n(x) = n_0 e^{-\left(\frac{x}{L/2}\right)^2} \text{ for } |x| < L/2, \text{ and } n = 0 \text{ for } |x| > L/2.$$

Substituting $\tilde{B}(x,t)$ into equation (6), we obtain:

$$\frac{\partial \tilde{b}}{\partial t} = D\frac{\partial^2 \tilde{b}}{\partial x^2} - \frac{icB_\perp}{4\pi e n_0} e^{\left(\frac{x}{L/2}\right)^2}\left[\frac{\partial^2 \tilde{b}}{\partial x^2} + \frac{2x}{(L/2)^2}\frac{\partial \tilde{b}}{\partial x}\right] - \frac{\tilde{B}_0}{\tau_R} e^{-t/\tau_R}. \qquad (7)$$

We introduce dimensionless variables $x = \frac{L}{2}\xi$ and $t = \tau\tau_D$, where:

$$\tau_D = \frac{\left(\frac{L}{2}\right)^2}{D}.$$

Define:

$$\alpha = \frac{\tau_D}{\tau_R}, \text{ and } \beta = \frac{D_H}{D},$$

where:

$$D_H = \frac{cB_\perp}{4\pi n_0 e}, \text{ and } D = \frac{c^2\eta}{4\pi}.$$

Equation (7) in dimensionless form becomes:

$$\frac{\partial \tilde{b}}{\partial \tau} = \frac{\partial^2 \tilde{b}}{\partial \xi^2} - i\beta e^{\xi^2}\left[\frac{\partial^2 \tilde{b}}{\partial \xi^2} + 2\xi \frac{\partial \tilde{b}}{\partial \xi}\right] - \tilde{B}_0\, \alpha e^{-\alpha\tau}. \qquad (8)$$

The initial condition is:

$$\tilde{b}(\xi, \tau = 0) = 0 \text{ for } 1 \geq \xi \geq 0. \qquad (9)$$

The boundary conditions are:

$$\tilde{b}(\xi = 1, \tau) = 0 \text{ for } \tau > 0, \text{ and}$$

$$\frac{\partial}{\partial \xi}\tilde{b}(\xi = 0, \tau) = 0 \text{ for } \tau > 0. \qquad (10)$$

$\tilde{b}(\xi, \tau)$ is an even function of ξ and need only to be determined in the domain $0 \leq \xi \leq L/2$.

A. The case of plasma of uniform density

For $n(x) = n_0$, equation (8) reduces to a simple diffusion equation:

$$\frac{\partial \tilde{b}}{\partial \tau} = \tilde{D}\frac{\partial^2 \tilde{b}}{\partial \xi^2} - \tilde{B}_0\, \alpha e^{-\alpha\tau}, \tag{11}$$

where the complex diffusion coefficient is defined as $\tilde{D} = (1 - i\beta)$. The analytical solution [4] to equation (11), satisfying equations (9) and (10), is:

$$\tilde{b}(\xi,\tau) = -\tilde{B}_0 \sum_{n=0}^{\infty} \frac{2(-1)^n}{k_n\left(1 - \tilde{D}\frac{k_n^{\,2}}{\alpha}\right)}\left(e^{-\tilde{D}k_n^2\tau} - e^{-\alpha\tau}\right)\cos k_n\xi\,, \tag{12}$$

where:

$$k_n = (2n+1)\frac{\pi}{2}.$$

These solutions are complex quantities, and are separated into real and imaginary parts. The real and imaginary parts of equation (12) are given by:

$$b_y(\xi,\tau) = -B_{0z}\sum_{n=0}^{\infty}\frac{2(-1)^n}{k_n\left[\left(1-\frac{k_n^{\,2}}{\alpha}\right)^2 + \left(\frac{\beta k_n^{\,2}}{\alpha}\right)^2\right]}\left\{e^{-k_n^2\tau}\left[\frac{\beta k_n^{\,2}}{\alpha}\cos(\beta k_n^{\,2}\tau) - \left(1-\frac{k_n^{\,2}}{\alpha}\right)\sin(\beta k_n^{\,2}\tau)\right] - \frac{\beta k_n^{\,2}}{\alpha}e^{-\alpha\tau}\right\}\cos k_n\xi\,, \text{ and} \tag{13}$$

$$b_z(\xi,\tau) = -B_{0z}\sum_{n=0}^{\infty}\frac{2(-1)^n}{k_n\left[\left(1-\frac{k_n^{\,2}}{\alpha}\right)^2 + \left(\frac{\beta k_n^{\,2}}{\alpha}\right)^2\right]}\left\{e^{-k_n^2\tau}\left[\frac{\beta k_n^{\,2}}{\alpha}\sin(\beta k_n^{\,2}\tau) + \left(1-\frac{k_n^{\,2}}{\alpha}\right)\cos(\beta k_n^{\,2}\tau)\right] - \left(1-\frac{k_n^{\,2}}{\alpha}\right)e^{-\alpha\tau}\right\}\cos k_n\xi \tag{14}$$

In the absence of the Hall diffusion coefficient, $\beta = 0$, $\tilde{D} = 1$, and $b_y = 0$, and equation (11) becomes:

$$\frac{\partial b_z}{\partial \tau} = \frac{\partial^2 b_z}{\partial \xi^2} - B_{0z}\alpha e^{-\alpha\tau} . \qquad (15)$$

Substituting $\beta = 0$ in equation (14), the solution to equation (15) becomes:

$$b_z(\xi,\tau) = -B_{0z}\sum_{n=0}^{\infty}\frac{2(-1)^n}{k_n\left(1 - \frac{k_n^{\ 2}}{\alpha}\right)}\left(e^{-k_n^2\tau} - e^{-\alpha\tau}\right)\cos k_n\xi \, . \qquad (16)$$

For a very short magnetic rise time, $\alpha \to \infty$ and the boundary condition at the vacuum-plasma interface becomes a step function in time.

For this case, equation (16) reduces to Armale's result [3]. The time evolution of equation (14) for $\xi = 0$, $\alpha = 20$, with $\beta = 10$, $\beta = 50$, and $\beta = 10^3$ are presented by figures 2 to 4 respectively. The oscillation frequency for these figures in dimensionless form is given by $\tilde{\omega}_n = \omega_n \tau_D$, with

$$\frac{\beta k_n^{\ 2}}{\tau_D} = \omega_{e\perp}\left(\frac{ck_n}{\omega_p}\right)^2 = \omega_n ,$$

where

$$\omega_{e\perp} = \frac{eB_\perp}{mc}, \text{ and } K_n = \frac{2}{L}k_n .$$

ω_n is known as a whistler frequency [8].

As shown in figures, the amplitude increases drastically as β drops from 10^3 to 10. Therefore, as β decreases, so does the oscillation frequency, but the amplitude of oscillation increases. Using equation (13), similar results obtain for b_y component.

This effect is due to the imaginary part of the complex diffusion coefficient containing the Hall resistivity. The exponential decaying parts of the components, b_y and b_z, are due to the real part of the complex diffusion coefficient.

This part is the usual collisional resistivity that determines the rate of the dissipative collisional diffusion. When the Hall resistivity is much larger than the collisional resistivity, i.e., $\beta = \eta_H/\eta \gg 1$, where

$$\eta_H = \frac{B_\perp}{n_0 ec},$$

the rate of the magnetic field penetration is determined by the Hall resistivity.

The experiment [1] shows that the average decay time is determined by the whistler time. They show for a magnetic field of 100 G, an observable penetration time of 17 ns, which corresponds to a frequency of 60 MHz. The whistler time and the diffusion time are scaled as:

$$\tau_\omega = \frac{4\pi\left(\frac{L}{2}\right)^2}{c^2\eta_H}, \text{ and } \tau_D = \frac{4\pi\left(\frac{L}{2}\right)^2}{c^2\eta}.$$

The ratio of $\tau_\omega/\tau_D = 1/\beta \ll 1$. For $\beta = 10^3$ and $\tau_D = 25\,\mu s$, the whistler time $\tau_\omega = 25$ ns, close to Armale's result [7]. From the plot of b_z, it is evident that this field should decay to zero in a time average equal to the whistler time. For $n = 2$, the time average integration of equation (14) approaches zero in a time scale $\langle\tau\rangle = 12$ ns, close to the observed time [1]. This clearly shows that the fast penetration time observed in the experiments [1] is due to the whistler time.

B. The case of plasma of non-uniform density

We now consider the numerical solution to equation (8) subject to initial and boundary conditions: equations (9) and (10). Numerical values for:

$$\beta = \frac{D_H}{D} = \omega_\perp \tau_{ei},$$

where

$$\omega_\perp = \frac{eB_\perp}{mc}, \text{ and } \alpha = \frac{\tau_D}{\tau_R} \gg 1,$$

are determined by the following assumed values: an electron temperature of $T_e = 10$ ev, electron density of $n_0 = 3 \times 10^{11}$ cm^{-3}, collision time of $\tau_{ei} = 10^{-6}$ sec, $B_\perp = 100$ G, $\beta \approx 10^3$, and $\alpha = 20$. We have also assumed that the rise time of the external magnetic field, τ_R, is much less than the diffusion time, τ_D.
Figures 4 and 5 represent the time evolution of b_z at locations, $\xi = 0$ and $\xi = 0.9$, with a constant density n_0 and a Gaussian

variable density given by $n(\xi) = n_0 e^{-\xi^2}$ respectively. As shown, the amplitude for $\xi = 0$ is the largest with respect to other amplitudes at other locations for the same value of τ. Figure 5 shows that the magnetic field amplitude grows as it penetrates from a low to a high-density region. A similar result is also obtained for b_y. The rate of growth of the magnetic field amplitude in the high-density region could be accounted for by the energy transport equation:

$$S = \frac{cB^2\upsilon_{ph}}{4\pi}, \text{ and}$$

the whistler dispersion relation:

$$\upsilon_{ph} = \frac{c^2\omega_{e\perp}K_n}{\omega_p^{\,2}} \propto \frac{1}{n(x)}.$$

S represents energy flow and υ_{ph} is whistler phase velocity. It is clear that in the high-density region, the phase velocity of the whistler waves that carries the field lines is reduced. This reduction causes the amplitude of the field inside the slab to increase in order to conserve power in the energy transport equation. A simpler density profile case has been carried out by Armale [7]. His results are similar to ours.

III. Conclusion

We have found that for a one-dimensional slab model of a neutralized ion beam of a constant density, when the Hall term is introduced into the linear diffusion equation, the magnetic field penetration is enhanced, if the background magnetic field has a component in the direction of penetration. It is shown that the Hall resistivity enables the magnetic field to penetrate as a whistler wave. We also found that the time average of the magnetic field due to the plasma current goes to zero on the whistler time scale rather than the diffusion time scale.

Furthermore, a numerical solution is obtained for the diffusion equation using a Gaussian density profile. We have found that the diffusion of the magnetic field from the low to high-density regions into the slab results in an increase of the magnetic field amplitude at a slower rate.

Acknowledgments

Special gratitude is due Professors Norman Rostoker at University of California, Irvine, Edwin L. Woollett and Alfred Leung at California State University, Long Beach for their supports and informative discussions in this topic.

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